SECTION 1.5.

PROBLEM 44. The matrix is already in reduced echelon form. It corresponds to a system of 3 equations in 4 unknowns, which we denote by x_1, x_2, x_3, and x_4. The leading 1's are in columns 1, 2, and 4. Hence, the leading variables are x_1, x_2, and x_4. The variable x_3 is a free variable. The equations are

$$x_1 - x_3 = -1$$
$$x_2 + 2x_3 = 1$$
$$x_4 = 1$$

Thus, we have:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 + 1x_3 \\ 1 - 2x_3 \\ 0 + 1x_3 \\ 1 + 0x_3 \end{bmatrix}$$

where x_3 is arbitrary. We can then describe the solutions in vector form as follows:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

where x_3 is arbitrary.

PROBLEM 57. First we calculate $P\mathbf{x} =$

$$\begin{bmatrix} .70 & .15 & .30 \\ .20 & .80 & .20 \\ .10 & .05 & .50 \end{bmatrix} \begin{bmatrix} 150,000 \\ 100,000 \\ 50,000 \end{bmatrix} = \begin{bmatrix} 105,000 + 15,000 + 15,000 \\ 30,000 + 80,000 + 10,000 \\ 15,000 + 5,000 + 25,000 \end{bmatrix} = \begin{bmatrix} 135,000 \\ 120,000 \\ 45,000 \end{bmatrix}$$

This represents the state vector after one year has passed.

Next, we calculate $P^2\mathbf{x}$. This is the same as $P(P\mathbf{x})$ by the associative law of multiplication. Using the above calculation, we obtain $P^2\mathbf{x} = P(P\mathbf{x}) =$

$$\begin{bmatrix} .70 & .15 & .30 \\ .20 & .80 & .20 \\ .10 & .05 & .50 \end{bmatrix} \begin{bmatrix} 135,000 \\ 120,000 \\ 45,000 \end{bmatrix} = \begin{bmatrix} 94,500 + 18,000 + 13,500 \\ 27,000 + 96,000 + 9,000 \\ 13,500 + 6,000 + 22,500 \end{bmatrix} = \begin{bmatrix} 126,000 \\ 132,000 \\ 42,000 \end{bmatrix}$$

This represents the state vector after two years have passed.

SECTION 1.6

PROBLEM 2. We have

$$(FE)D = \left(\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}3 & 6\\2 & 3\end{bmatrix}\right)\begin{bmatrix}2 & 1\\1 & 4\end{bmatrix} = \begin{bmatrix}5 & 9\\5 & 9\end{bmatrix}\begin{bmatrix}2 & 1\\1 & 4\end{bmatrix} = \begin{bmatrix}19 & 41\\19 & 41\end{bmatrix}$$

On the other hand, we have

$$F(ED) = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\left(\begin{bmatrix}3 & 6\\2 & 3\end{bmatrix}\begin{bmatrix}2 & 1\\1 & 4\end{bmatrix}\right) = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}12 & 27\\7 & 14\end{bmatrix} = \begin{bmatrix}19 & 41\\19 & 41\end{bmatrix}$$

This verifies that $(FE)D = F(ED)$.

PROBLEM 12. We have $(EF)\mathbf{v} =$

$$\left(\begin{bmatrix}3 & 6\\2 & 3\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\right)\begin{bmatrix}-3\\3\end{bmatrix} = \begin{bmatrix}9 & 9\\5 & 5\end{bmatrix}\begin{bmatrix}-3\\3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

PROBLEM 20. First we compute $D\mathbf{v}$, obtaining

$$\begin{bmatrix}2 & 1\\1 & 4\end{bmatrix}\begin{bmatrix}-3\\3\end{bmatrix} = \begin{bmatrix}-3\\9\end{bmatrix}$$

Then $||D\mathbf{v}|| = \sqrt{(-3)^2 + 9^2} = \sqrt{90} = 3\sqrt{10}$.

PROBLEM 26. We can use the distributive law to obtain

$$(A-B)(A+B) = (A-B)A + (A-B)B = (AA-BA) + (AB-BB) = A^2 - B^2 + AB - BA.$$

This is equal to $A^2 - B^2$ if and only if $AB - BA$ is the 2×2 zero matrix. Thus, the statement in question can hold if and only if $AB = BA$. There are counterexamples to the statement. Just choose any two matrices A and B such that $AB \neq BA$. For example, one could choose

$$A = \begin{bmatrix}2 & 3\\4 & 5\end{bmatrix}, \quad B = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

One easily checks that $AB \neq BA$ and so the statement in this question is false for this choice of A and B.

PROBLEM 42(a). The equation is $A^T + B = C$. This implies that

$$A^T \quad = \quad C - B \quad = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \quad - \quad \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \quad = \quad \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

It follows that

$$A \quad = \quad (A^T)^T \quad = \quad \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

SECTION 1.7.

PROBLEM 2. Notice that $\boldsymbol{v}_3 = 2\boldsymbol{v}_1$. Hence $2\boldsymbol{v}_1 + (-1)\boldsymbol{v}_3 = \mathbf{0}$. Therefore the set $\{\boldsymbol{v}_1, \boldsymbol{v}_3\}$ is a linearly dependent set.

PROBLEM 4. Consider the matrix A = $[\boldsymbol{v}_2 \quad \boldsymbol{v}_3]$. We reduce this matrix to a matrix in echelon form:

$$A \quad = \quad \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

The matrix A has rank 2 and so the set $\{\boldsymbol{v}_2, \boldsymbol{v}_3\}$ is a linearly independent set.

Much more simply, note that neither vector is a scalar multiple of the other vector. Since there are just two vectors in the set, it is a linearly independent set.

PROBLEM 6. The set $\{\boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ must be a linearly dependent set because the matrix A = $[\boldsymbol{v}_2 \quad \boldsymbol{v}_3 \quad \boldsymbol{v}_4]$ cannot have rank 3. The matrix A is a 2×3 matrix and so rank(A) ≤ 2.

We can express $\boldsymbol{v}_4$ as a linear combination of $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ by solving the vector equation

$$x\boldsymbol{v}_2 + y\boldsymbol{v}_3 = \boldsymbol{v}_4$$

The corresponding augmented matrix is

$$\begin{bmatrix} 2 & 2 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

and this is row-equivalent to

$$\begin{bmatrix} 1 & 1 & 1/2 \\ 3 & 4 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1/2 \\ 0 & 1 & -1/2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1/2 \end{bmatrix}$$

Thus, we have $x = 1, y = -1/2$ as the solution to the above vector equation. Hence

$\boldsymbol{v}_4 = 1\boldsymbol{v}_2 - 1/2\boldsymbol{v}_3$

PROBLEM 12. Consider the matrix A $= [\boldsymbol{u}_1 \quad \boldsymbol{u}_2 \quad \boldsymbol{u}_4]$. We will reduce this matrix to a matrix in echelon form.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 4 \\ -1 & -3 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & -4 \\ 0 & -1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4/3 \\ 0 & -1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4/3 \\ 0 & 0 & 16/3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4/3 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus rank(A) $= 3$. By the criterion for linear independence, the set of vectors $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_4\}$ is a linearly independent set.

PROBLEM 14. The set of vectors $\{\boldsymbol{u}_0, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\}$ is a linearly dependent set. This is because the matrix A which has these four vectors as columns is a 3×4 matrix. It is impossible for rank(A) to be equal to 4. Thus, there will be nontrivial solutions to the vector equation

$$x\boldsymbol{u}_0 + y\boldsymbol{u}_2 + z\boldsymbol{u}_3 + w\boldsymbol{u}_4 = \mathbf{0}$$

where

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The corresponding augmented matrix is:

$$\begin{bmatrix} 1 & 2 & -1 & 4 & 0 \\ 0 & 1 & 4 & 4 & 0 \\ 0 & -3 & 3 & 0 & 0 \end{bmatrix}$$

Row-reduction gives the following sequence of matrices:

$$\begin{bmatrix} 1 & 2 & -1 & 4 & 0 \\ 0 & 1 & 4 & 4 & 0 \\ 0 & -3 & 3 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -9 & -4 & 0 \\ 0 & 1 & 4 & 4 & 0 \\ 0 & -3 & 3 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -9 & -4 & 0 \\ 0 & 1 & 4 & 4 & 0 \\ 0 & 0 & 15 & 12 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & -9 & -4 & 0 \\ 0 & 1 & 4 & 4 & 0 \\ 0 & 0 & 1 & 4/5 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 16/5 & 0 \\ 0 & 1 & 4 & 4 & 0 \\ 0 & 0 & 1 & 4/5 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 16/5 & 0 \\ 0 & 1 & 0 & 4/5 & 0 \\ 0 & 0 & 1 & 4/5 & 0 \end{bmatrix}$$

The solutions to the above vector equation are given by
$x = -16/5w, \quad y = -4/5w, \quad z = -4/5w$, where w is arbitrary. Taking $w = -1$ gives the solution $x = 16/5, y = 4/5, z = 4/5, w = -1$. That is,

$16/5\boldsymbol{u}_0 + 4/5\boldsymbol{u}_2 + 4/5\boldsymbol{u}_3 - 1\boldsymbol{u}_4 = \mathbf{0}$

using this equation, we can express $\boldsymbol{u}_4$ as a linear combination of the other vectors as follows:

$$\boldsymbol{u}_4 = 16/5\boldsymbol{u}_0 + 4/5\boldsymbol{u}_2 + 4/5\boldsymbol{u}_3.$$

PROBLEM 18. The matrix C is 2×2. We perform row-reduction on C as follows:

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We see that rank(C) = 2 and so C is nonsingular.

PROBLEM 22. The matrix F is 3×3. Multiplying row 2 by 1/3, we obtain:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \end{bmatrix}$$

This matrix is in echelon form. Thus, we see that rank(F) = 3 and so F is nonsingular.

PROBLEM 24. The matrix E is 3×3. Notice that the three columns are linearly dependent because

$$1 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad + \quad 0 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad + \quad 0 \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \quad = \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the three columns of E form a linearly dependent set, it follows that rank(E) $\neq 3$ and so E is a singular matrix.

Now we will solve the matrix equation EX = $\mathbf{0}$, where

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

.

We reduce the augmented matrix $[E|\ \mathbf{0}]$ as follows:

$$E = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(Remark: Obviously, it is not really necessary to always include the 4-th column in this case. Since we are considering a homogeneous system of equations, the 4th column will consist of 0's and those won't change in the process of row-reduction.)

The solutions to EX $= \mathbf{0}$ are given by $y = z = 0$, or in vector form,

$$X = c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

where c is arbitrary.

PROBLEM 38. The question concerns the matrix equation FX $= \boldsymbol{u}_1$, where

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

.

We will solve the matrix equation

$$FX = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

.

We will perform row-reduction on the corresponding augmented matrix:

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2/3 & 2/3 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -1/3 & -1/3 \\ 0 & 1 & 2/3 & 2/3 \\ 0 & 0 & 1 & -1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 & -2/3 \\ 0 & 1 & 2/3 & 2/3 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & -2/3 \\ 0 & 1 & 0 & 4/3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The solution to the matrix equation FX $= \boldsymbol{u}_1$ is $x = -2/3, y = 4/3, z = -1$. Thus, the equivalent vector equation has the same solution. That is,

$$\boldsymbol{u}_1 = -2/3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad + \quad 4/3 \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \quad - \quad \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

This expresses $\boldsymbol{u}_1$ as a linear combination of the columns of the matrix F.

ADDITIONAL QUESTIONS

QUESTION A. Let $A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 2 & 2 & 0 & 3 \end{bmatrix}$.

(a) We want to find all solutions to the matrix equation $AX = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and to express the answer in "vector form." That matrix equation is equivalent to a system of 2 equations in 5 unknowns. We denote the unknowns by $x_1,\ x_2,\ x_3,\ x_4$ and x_5. Then we have

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} .$$

We perform row-reduction on the augmented matrix $\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & 0 & 3 & 3 \end{bmatrix}$ to solve the stated matrix equation:

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & 0 & 3 & 3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Reintroducing the unknowns, we have the system of equations

$$\begin{cases} 0x_1 + 1x_2 + 1x_3 + 0x_4 + 0x_5 = 0 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 + 1x_5 = 1 \end{cases}$$

The leading variables are x_2 and x_5. The free variables are x_1, x_3 and x_4. The solutions are described by

$$x_1 = x_1, \qquad x_2 = -x_3, \qquad x_3 = x_3, \qquad x_4 = x_4, \qquad x_5 = 1,$$

where $x_1,\ x_3$, and x_4 are arbitrary. The solutions to the matrix equation are given by

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_3 \\ x_3 \\ x_4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 + 1x_1 + 0x_3 + 0x_4 \\ 0 + 0x_1 - 1x_3 + 0x_4 \\ 0 + 0x_1 + 1x_3 + 0x_4 \\ 0 + 0x_1 + 0x_3 + 1x_4 \\ 1 + 0x_1 + 0x_3 + 0x_4 \end{bmatrix}$$

where $x_1,\ x_3$ and x_4 are arbitrary. Thus, we can describe the solutions to the matrix equation $AX = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ in vector form as follows:

$$X = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad + \quad x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad + \quad x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad + \quad x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

where $x_1,\ x_3$ and x_5 are arbitrary.

(b) Since $X = U_1$ and $X = U_2$ are solutions to the matrix equation considered in part (a), we know that

$$AU_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad AU_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Now $V = 4U_1 + 6U_2$. We can calculate AV by the distributive law. We obtain

$$AV = A(4U_1 + 6U_2) = 4AU_1 + 6AU_2 = 4\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 6\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \end{bmatrix} \quad + \quad \begin{bmatrix} 6 \\ 18 \end{bmatrix} \quad = \quad \begin{bmatrix} 10 \\ 30 \end{bmatrix}$$

Hence AV can be determined and we have $AV = \begin{bmatrix} 10 \\ 30 \end{bmatrix}$.

(c) The matrix A is 2×5. We want AB to be a certain 2×3 matrix. Hence B must be a 5×3 matrix. We will write $B = [X \quad Y \quad Z]$, where X, Y, and Z are the three columns of B. They are 5×1 matrices. With this notation, we have $AB = [AX \quad AY \quad AZ]$. We want to have

$$AX = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad AY = \begin{bmatrix} 5 \\ 15 \end{bmatrix}, \qquad AZ = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The first matrix equation has already been solved in part (a). We just need one solution and so we will simply take $x_1 = x_3 = x_4 = 0$. To choose a suitable Y, we can simply multiply the X just chosen by 5. For the choice of Z, we can just take Z to be the 5-dimensional zero vector. Thus, one possible B is the following matrix:

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 5 & 0 \end{bmatrix}$$

QUESTION B. According to the hint, we should first solve the matrix equation $AX = \mathbf{0}$, where

$$A = P - I_3 = \begin{bmatrix} .70 & .15 & .30 \\ .20 & .80 & .20 \\ .10 & .05 & .50 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -.30 & .15 & .30 \\ .20 & -.20 & .20 \\ .10 & .05 & -.50 \end{bmatrix}$$

This last matrix is A. We will find the reduced echelon form E for A. The solutions to the matrix equations $AX = \mathbf{0}$ and $EX = \mathbf{0}$ will be the same. This is because the corresponding augmented matrix will have $\mathbf{0}$ as its final column. The elementary row operations will not change that final column and so there is no need to keep track of it.

$$\begin{bmatrix} -.30 & .15 & .30 \\ .20 & -.20 & .20 \\ .10 & .05 & -.50 \end{bmatrix}, \quad \begin{bmatrix} 1 & -.5 & -1 \\ .20 & -.20 & .20 \\ .10 & .05 & -.50 \end{bmatrix}, \quad \begin{bmatrix} 1 & -.5 & -1 \\ 0 & -.10 & .40 \\ 0 & .10 & -.40 \end{bmatrix},$$

$$\begin{bmatrix} 1 & -.5 & -1 \\ 0 & 1 & -4 \\ 0 & .10 & -.40 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus

$$E = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Reintroducing the variables a, b, and c gives the equations
$a - 3c = 0$
$b - 4c = 0$
$0a + 0b + 0c = 0$,
and the solutions are given by $a = 3c, b = 4c$, where c is arbitrary. Therefore, the solutions to $PX = X$ are given by

$$X = \begin{bmatrix} 3c \\ 4c \\ c \end{bmatrix}$$

We want $a + b + c = 300,000$. That is, $3c + 4c + c = 300,000$, which gives $c = 37,500$. Thus, the solution we want is:

$$X = \begin{bmatrix} 112,500 \\ 150,000 \\ 37,500 \end{bmatrix}.$$