SECTION 1.7.

PROBLEM 40. The problem is to find a solution to the following vector equation:

$$x \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad + \quad y \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad = \quad \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

This vector equation is equivalent to a system of 2 equations in the 2 unknowns x and y. The augmented matrix for this system is:

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 7 \end{bmatrix}$$

Row-reduction gives:

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 7 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -3 \end{bmatrix}$$

The solution is $x = 8, \quad y = -3$. Therefore

$$\begin{bmatrix} 2 \\ 7 \end{bmatrix} \quad = \quad 8 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad - \quad 3 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

SECTION 1.9.

PROBLEM 2.

$$AB = \begin{bmatrix} 3 & 10 \\ 2 & 10 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -.2 & .3 \end{bmatrix} = \begin{bmatrix} 3-2 & -3+3 \\ 2-2 & -2+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -1 \\ -.2 & .3 \end{bmatrix} \begin{bmatrix} 3 & 10 \\ 2 & 10 \end{bmatrix} = \begin{bmatrix} 3-2 & 10-10 \\ -.6+.6 & -2+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

PROBLEM 4.

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2-2 & 1 & 0 \\ 3-8+5 & 4-4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2+2 & 1 & 0 \\ 5-8+3 & -4+4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

PROBLEM 8. The system of equations in this question is equivalent to the matrix equation

$$BX = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

where B is as in problem 4. We have $B^{-1} = A$ where A is as in problem 4. The solution to the given matrix equation is:

$$X = B^{-1} \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = A \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4+3 \\ 6+12+2 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 20 \end{bmatrix}$$

PROBLEM 14. To find the inverse of A, we reduce the augmented matrix $[A|I_2]$ to its reduced echelon form:

$$[A|I_2] = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 6 & 7 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3/2 & 1/2 & 0 \\ 6 & 7 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3/2 & 1/2 & 0 \\ 0 & -2 & -3 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3/2 & 1/2 & 0 \\ 0 & 1 & 3/2 & -1/2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -7/4 & 3/4 \\ 0 & 1 & 3/2 & -1/2 \end{bmatrix}$$

Therefore, we see that

$$A^{-1} = \begin{bmatrix} -7/4 & 3/4 \\ 3/2 & -1/2 \end{bmatrix}$$

PROBLEM 16. To find A^{-1}, we reduce the augmented matrix $[A|I_3]$ to its reduced echelon form:

$$[A|I_3] = \begin{bmatrix} -1 & -2 & 11 & 1 & 0 & 0 \\ 1 & 3 & -15 & 0 & 1 & 0 \\ 0 & -1 & 5 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & -11 & -1 & 0 & 0 \\ 1 & 3 & -15 & 0 & 1 & 0 \\ 0 & -1 & 5 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & -11 & -1 & 0 & 0 \\ 0 & 1 & -4 & 1 & 1 & 0 \\ 0 & -1 & 5 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & -11 & -1 & 0 & 0 \\ 0 & 1 & -4 & 1 & 1 & 0 \\ 0 & -1 & 5 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -3 & -3 & -2 & 0 \\ 0 & 1 & -4 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 5 & 5 & 4 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Therefore

$$A^{-1} = \begin{bmatrix} 0 & 1 & 3 \\ 5 & 5 & 4 \\ 1 & 1 & 1 \end{bmatrix}$$

PROBLEM 18. To find A^{-1}, we reduce the augmented matrix $[A|I_3]$ to its reduced echelon form:

$$[A|I_3] = \begin{bmatrix} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 2 & 7 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -7 & 1 & -3 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 2 & 7 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -7 & 1 & -3 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -2 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & -7 & 1 & -3 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -7 & 1 & -3 & 0 \\ 0 & 1 & 0 & 0 & -7 & 4 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 1 & 11 & -7 \\ 0 & 1 & 0 & 0 & -7 & 4 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{bmatrix}$$

Therefore

$$A^{-1} = \begin{bmatrix} 1 & 11 & -7 \\ 0 & -7 & 4 \\ 0 & 2 & -1 \end{bmatrix}$$

PROBLEM 24. As in example 4, we find A^{-1} as follows. We have

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}$$

and so $\Delta = \det(A) = (-1) \times 1 - 2 \times 3 = -7$. Thus

$$A^{-1} = \frac{1}{-7} \begin{bmatrix} 1 & -3 \\ -2 & -1 \end{bmatrix} \quad = \quad \begin{bmatrix} -1/7 & 3/7 \\ 2/7 & 1/7 \end{bmatrix}$$

PROBLEM 40. We have $Q = B^{-1}A$ and so

$$Q^{-1} = A^{-1}B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 4 & 2 \end{bmatrix}$$

PROBLEM 42. We have

$$Q^{-1} = (B^{-1})^{-1} = B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

PROBLEM 50. In this problem, A is a 3×3 nonsingular matrix. Therefore A is invertible. Also, we have

$$A^2 = AB + 2A = AB + 2AI_3 = AB + A(2I_3) = A(B + 2I_3)$$

by the distributive law. Multiplying by A^{-1}, we obtain

$$A^{-1}A^2 = A^{-1}A(B+2I_3) = I_3(B+2I_3) = B+2I_3$$

Now $A^{-1}A^2 = A^{-1}(AA) = (A^{-1}A)A = I_3A = A$ and so the above becomes

$$A = B + 2I_3 = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \\ -1 & 4 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 5 & 2 \\ -1 & 4 & 3 \end{bmatrix}$$

PROBLEM 51. We have

$$(A^{-1}B)^{-1}(C^{-1}A)^{-1}(B^{-1}C)^{-1} = (B^{-1}A)(A^{-1}C)(C^{-1}B)$$

$$= B^{-1}(AA^{-1})(CC^{-1})B = B^{-1}I_3I_3B = B^{-1}B = I_3$$

PROBLEM A: We are told that A is a 4×4 matrix and that

$$A\begin{bmatrix} 2 \\ 3 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$

We are told nothing else about A.

(a) The vectors $V_1,\ V_2,\ V_3$, and V_4 are the four columns of the matrix A. If $\mathbf{b}$ is any vector in $\mathbf{R}^4$, then the matrix equation $AX = \mathbf{b}$ is equivalent to the vector equation

$$x_1V_1 + x_2V_2 + x_3V_3 + x_4V_4 \ = \ \mathbf{b} \ .$$

We will take $\mathbf{b}$ to be the zero-vector in $\mathbf{R}^4$. The given information about A then provides us with one solution to the matrix equation and hence to the equivalent vector equation, namely $x_1 = 2,\ x_2 = 3,\ x_3 = 0,\ x_4 = 5$. Therefore, we can say that

$$2V_1 \ + \ 3V_2 \ + \ 0V_3 \ + \ 5V_4 \ = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$

It follows that $3V_2 \;=\; -\,2V_1 - 5V_4$. Therefore,

$$V_2 \;=\; -\frac{2}{3}V_1 \;-\; \frac{5}{3}V_4 \;.$$

Thus, we can take $a = -\frac{2}{3}$ and $b = -\frac{5}{3}$.

(b) The matrix A is a 4×4 matrix. As stated above, the matrix equation

$$AX = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

has a nontrivial solution, namely

$$X = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 5 \end{bmatrix}$$

Therefore, the 4×4 matrix A cannot be nonsingular. It must be a singular matrix. Therefore, $rank(A) < 4$. Suppose that $\mathbf{b}$ is a vector in $\mathbf{R}^4$. We can conclude two things about the matrix equation $AX = \mathbf{b}$ from the fact that $rank(A) < 4$. First of all, for some choices of the vector $\mathbf{b}$, that matrix equation will have no solutions. Secondly, if the matrix equation $AX = \mathbf{b}$ has a solution, then it will have infinitely many solutions. We cannot determine which case occurs for a given nonzero vector $\mathbf{b}$ without knowing more about the matrix A. In particular, the matrix equation

$$AX \;=\; \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

may have either no solutions or infinitely many solutions.

PROBLEM B: The equation $PX \;=\; \lambda X$ is equivalent to the equation $AX \;=\; \mathbf{0}$, where $A = P - \lambda I_3$. We consider $\lambda = .4, \;\; \lambda = .5, \;$ and $\lambda = .6$ separately. Note that

$$A = \begin{bmatrix} .7 & .15 & .3 \\ .2 & .8 & .2 \\ .1 & .05 & .5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} .7 & .15 & .3 \\ .2 & .8 & .2 \\ .1 & .05 & .5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} .7-\lambda & .15 & .3 \\ .2 & .8-\lambda & .2 \\ .1 & .05 & .5-\lambda \end{bmatrix}$$

We will use this for the values $\lambda = .4, \quad \lambda = .5,$ and $\lambda = .6$. For each value of λ, we consider the matrix equation $AX = \mathbf{0}$. We can solve this by performing row-reduction on the augmented matrix $[A|\ \mathbf{0}]$. We will omit the last column $\mathbf{0}$ because that will not change through the row-reduction process.

$\lambda = .4$. Then row-reduction gives

$$A = \begin{bmatrix} .3 & .15 & .3 \\ .2 & .4 & .2 \\ .1 & .05 & .1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1/2 & 1 \\ .2 & .4 & .2 \\ .1 & .05 & .1 \end{bmatrix}, \quad \begin{bmatrix} 1 & .5 & 1 \\ 0 & .3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & .5 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix E is the reduced echelon form for A. The matrix equation $AX = \mathbf{0}$ is equivalent to the matrix equation $EX = \mathbf{0}$. Here we will write

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

so that the unknowns are x, y and z. Clearly, x and y are the leading variables, z is the free variable, and the solutions are given by

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ 0 \\ z \end{bmatrix} = z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

One nontrivial solution is

$$X = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = .5$. Then row-reduction gives

$$A = \begin{bmatrix} .2 & .15 & .3 \\ .2 & .3 & .2 \\ .1 & .05 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3/4 & 3/2 \\ 2 & 3 & 2 \\ 1 & 1/2 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3/4 & 3/2 \\ 0 & 3/2 & -1 \\ 0 & -1/4 & -3/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3/4 & 3/2 \\ 0 & 1 & -2/3 \\ 0 & -1/4 & -3/2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3/4 & 3/2 \\ 0 & 1 & -2/3 \\ 0 & 0 & -5/3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3/4 & 3/2 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}$$

This last matrix is in echelon form and so we see that A has rank 3. The matrix equation $AX = \mathbf{0}$ has no nontrivial solutions. Hence, the equation $PX = .5X$ has no solutions except for the trivial solution $X = \mathbf{0}$.

$\lambda = .6$. Then row-reduction gives

$$A = \begin{bmatrix} .1 & .15 & .3 \\ .2 & .2 & .2 \\ .1 & .05 & -.1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3/2 & 3 \\ 1 & 1 & 1 \\ 1 & 1/2 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3/2 & 3 \\ 0 & -1/2 & -2 \\ 0 & -1 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3/2 & 3 \\ 0 & 1 & 4 \\ 0 & -1 & -4 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix E is in reduced echelon form. The matrix equation $AX = \mathbf{0}$ is equivalent to the matrix equation $EX = \mathbf{0}$. We denote the unknowns by x, y, and z. The solutions are given by

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3z \\ -4z \\ z \end{bmatrix} = z \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$$

One nontrivial solution is

$$X = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$$