SECTION 3.3.

PROBLEM 22. The null space of a matrix A is: $\mathcal{N}(A) = \{X : AX = \mathbf{0}\}$.

Here are the calculations of AX for $X = \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$, and $\boldsymbol{e}$.

$$A\boldsymbol{a} = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2-2 \\ 3-3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A\boldsymbol{b} = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4-6 \\ 6-9 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

$$A\boldsymbol{c} = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4+4 \\ -6+6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A\boldsymbol{d} = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2+0 \\ 3+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$A\boldsymbol{e} = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore, the vectors $\boldsymbol{a}, \boldsymbol{c}$, and $\boldsymbol{e}$ are in $\mathcal{N}(A)$, but the vectors $\boldsymbol{b}$ and $\boldsymbol{d}$ are not.

PROBLEM 24. Here are the calculations of AX for $X = \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}$, and $\boldsymbol{z}$.

$$A\boldsymbol{v} = \begin{bmatrix} -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2+2+0 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}, \quad A\boldsymbol{w} = \begin{bmatrix} -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0-1+1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix},$$

$$A\boldsymbol{x} = \begin{bmatrix} -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2+1-1 \end{bmatrix} = \begin{bmatrix} -2 \end{bmatrix}, \quad A\boldsymbol{y} = \begin{bmatrix} -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4-2+2 \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix}$$

$$A\boldsymbol{z} = \begin{bmatrix} -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2+0+2 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

Hence, the vectors $\boldsymbol{v}, \boldsymbol{w}$, and $\boldsymbol{z}$ are in $\mathcal{N}(A)$, but the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are not.

PROBLEM 34. The null space of a matrix A is: $\mathcal{N}(A) = \{X : AX = \mathbf{0}\}$. We can find a simple algebraic specification for $\mathcal{N}(A)$ by reducing A to its reduced echelon form:

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 5 \\ 1 & 0 & 7 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix E is in reduced echelon form. Since $\mathcal{N}(A) = \mathcal{N}(E)$, we have the following algebraic specification for $\mathcal{N}(A)$:

$$x_1 + 7x_3 = 0, \quad x_2 + 3x_3 = 0$$

To find an algebraic specification for $\mathcal{R}(A)$, we consider the augmented matrix $[A|\boldsymbol{b}]$, where

$$\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Row-reduction gives:

$$[A|\boldsymbol{b}] = \begin{bmatrix} 1 & -2 & 1 & b_1 \\ 2 & -3 & 5 & b_2 \\ 1 & 0 & 7 & b_3 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 & 1 & b_1 \\ 0 & 1 & 3 & b_2 - 2b_1 \\ 0 & 2 & 6 & b_3 - b_1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 & 1 & b_1 \\ 0 & 1 & 3 & \\ 0 & 0 & 0 & (b_3 - b_1) - 2(b_2 - 2b_1) \end{bmatrix}$$

The last matrix is in echelon form. The matrix equation $AX = \boldsymbol{b}$ can be solved if and only if $(b_3 - b_1) - 2(b_2 - 2b_1) = 0$. This simplifies to

$$3b_1 - 2b_2 + b_3 = 0$$

This equation is the algebraic specification for $\mathcal{R}(A)$.

PROBLEM 36. The null space of a matrix A is: $\mathcal{N}(A) = \{X : AX = \mathbf{0}\}$. We can find a simple algebraic specification for $\mathcal{N}(A)$ by reducing A to its reduced echelon form:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, we see that A has rank 3 and so is a nonsingular matrix. The matrix equation $AX = \mathbf{0}$ has only one solution, namely $X = \mathbf{0}$. Hence

$$\mathcal{N}(A) = \{\mathbf{0}\}$$

Also, since A is nonsingular, it follows that for every vector $\boldsymbol{b}$ in $\boldsymbol{R}^3$, the matrix equation $AX = \boldsymbol{b}$ has at least one solution (in fact, exactly one solution). Therefore,

$$\mathcal{R}(A) = \boldsymbol{R}^3$$

SECTION 3.4.

PROBLEM 2. We solve this system of equations by Gauss-Jordan elimination. Sunstracting the 2nd equation from the 1st equation gives the following equivalent system of equations:

$$\begin{aligned} x_1 + x_3 + 2x_4 &= 0 \\ x_2 - 2x_3 - x_4 &= 0 \end{aligned}$$

Thus, the leading variables are x_1 and x_2. The free variables are x_3 and x_4. The solutions to the given system of equations are given in vector form by:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 - 2x_4 \\ 2x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

where x_3 and x_4 are arbitrary numbers. In this question, W is a subspace of $\boldsymbol{R}^4$ and has the following basis:

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

PROBLEM 4. We just have a system of 1 equation in the 4 unknowns x_1, x_2, x_3 and x_4. Note that x_1 is the leading variable and x_2, x_3 and x_4 are the free variables. Here is a description of the solutions in vector form:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - x_3 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

In this question, W is a subspace of $\boldsymbol{R}^4$ and has the following basis:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

PROBLEM 10b. The vector

$$\boldsymbol{x} = \begin{bmatrix} 0 \\ 3 \\ 2 \\ -1 \end{bmatrix}$$

is in W because $x_1 = 0, x_2 = 3, x_3 = 2, x_4 = -1$ is a solution to the homogeneous system of equations given in exercise 2. That is,

$$0 + 3 - 2 + (-1) = 0$$
$$3 - 2(2) - (-1) = 0$$

In order to express $\boldsymbol{x}$ as a linear combination of the basis vectors found in exercise 2, we just consider the vector equation

$$a \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \\ -1 \end{bmatrix}$$

We get the solution $\boldsymbol{x}$ by letting $a = x_3 = 2$ and $b = x_4 = -1$. Thus,

$$2 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \\ -1 \end{bmatrix}$$

PROBLEM 10c. The vector

$$\boldsymbol{x} = \begin{bmatrix} 7 \\ 8 \\ 3 \\ 2 \end{bmatrix}$$

is not in W because $x_1 = 7, x_2 = 8, x_3 = 3, x_4 = 2$ is not a solution to the system of equations given in exercise 2. The first equation in the system is not satisfied because

$$7 + 8 - 3 + 2 = 14 \neq 0$$

PROBLEM 11b. We can find a basis for the null space of A by finding the reduced echelon form for A. We do the row-reduction as follows:

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 3 & 5 & 8 & -2 \\ 1 & 1 & 2 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & -1 & -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The last matrix E is the reduced echelon form for A. The solutions to $AX = \mathbf{0}$ are the same as the solutions to $EX = \mathbf{0}$. Denoting the unknowns by x_1, x_2, x_3, x_4, the solutions to $EX = \mathbf{0}$ are described by the equations $x_1 = -x_3 - x_4, \quad x_2 = -x_3 + x_4$ where x_3 and x_4 are arbitrary numbers. In vector form, the solutions are given by

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 - x_4 \\ -x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

In this question, $W = \mathcal{N}(A)$ is a subspace of $\boldsymbol{R}^4$ and has the following basis:

$$\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

PROBLEM 12b. We can find a basis for the null space of A by finding the reduced echelon form for A. We do the row-reduction as follows:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 3 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix E is the reduced echelon form for A. The solutions to $AX = \mathbf{0}$ are the same as the solutions to $EX = \mathbf{0}$. Denoting the unknowns by x_1, x_2, x_3, the solutions to $EX = \mathbf{0}$ are described by the equations $x_1 = -x_3, \quad x_2 = -x_3$ where x_3 is an arbitrary number. In vector form, the solutions are given by

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

In this question, $W = \mathcal{N}(A)$ is a subspace of $\boldsymbol{R}^3$ and has the following basis:

$$\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

PROBLEM 32. We first determine if the matrix A which has the three given vectors as its columns is singular or nonsingular:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & -3 \\ -2 & 2 & -3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 4 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1/2 \\ 0 & 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

The last matrix is in echelon form and so the rank of A is 3. This means that A is nonsingular. Thus, the set S is a linearly independent set consisting of three vectors and therefore S is a basis of $\boldsymbol{R}^3$.

PROBLEM 33. As in problem 33, we first determine if the matrix A which has the three given vectors as its columns is singular or nonsingular:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 5 & 3 \\ -2 & 2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 6 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2/3 \\ 0 & 6 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix is in echelon form. The rank of A is 2. Hence A is a singular matrix. The set S is a linearly dependent set. Hence S is not a basis for $\boldsymbol{R}^3$.

PROBLEM 34. The matrix A which has the four vectors in S as its columns is a 3×4 matrix. Hence, $\text{rank}(A) \leq 3$ and hence the rank of A cannot be four. The set S is a linearly dependent set and so S isn't a basis for $\boldsymbol{R}^3$.

PROBLEM 35. The set S consists of two vectors in $\boldsymbol{R}^3$. Those two vectors determine a plane π in $\boldsymbol{R}^3$ through the origin. It is clear that $Sp(S)$ is the set of vectors which lie on that plane π. Thus, $Sp(S) \neq \boldsymbol{R}^3$. Therefore S is not a spanning set for $\boldsymbol{R}^3$. Therefore S cannot be a basis for $\boldsymbol{R}^3$.

SECTION 3.5.

PROBLEM 10. Since neither vector in the set $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ is equal to scalar multiple of the other, the set S is a linearly independent set of two vectors in $\boldsymbol{R}^2$. Using part 3 of theorem 9, it follows that S is a basis for $\boldsymbol{R}^2$.

PROBLEM 11. Since neither vector in the set $S = \{\boldsymbol{u}_2, \boldsymbol{u}_3\}$ is equal to scalar multiple of the other, the set S is a linearly independent set of two vectors in $\boldsymbol{R}^2$. Hence S is a basis for $\boldsymbol{R}^2$.

PROBLEM 12. The set S contains three vectors in $\boldsymbol{R}^3$. This set S will be a basis for $\boldsymbol{R}^3$ if and only if the S is a linearly independent set. We test this by determining whether the matrix A which has the vectors in S as its columns is singular or nonsingular:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, A has rank 3 and so A is nonsingular. Therefore, S is a basis for $\boldsymbol{R}^3$.

PROBLEM 13. We are given a set S of three vectors in $\boldsymbol{R}^3$. We will consider the matrix A which has the three vectors in S as its columns. We determine the rank of A as follows:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix is in echelon form and so A has rank 2. This implies that S is a linearly dependent set. Therefore, S is not a basis for $\boldsymbol{R}^3$.

PROBLEM 22. First, we find the reduced echelon form for A:

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -5 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 & 0 \\ 2 & -5 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix}$$

The matrix E is the reduced echelon form for A. The solutions to the matrix equation $AX = \mathbf{0}$ are the same as the solutions to the matrix equation $EX = \mathbf{0}$. That is, $\mathcal{N}(A) = \mathcal{N}(E)$. If we denote the unknowns by x_1, x_2, x_3, then x_1 and x_2 are the leading variables and x_3 is a free variable. The solutions to $AX = \mathbf{0}$ are described by $x_1 - 2x_3 = 0, x_2 - x_3 = 0$. In vector form, the solutions are described by:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

A basis for $\mathcal{N}(A)$ is:

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Hence $\dim(\mathcal{N}(A)) = 1$ and so the nullity of A is 1. The rank of A is 2 because the reduced echelon form E for A has 2 nonzero rows.

PROBLEM 24. The null space of A is a subspace of $\boldsymbol{R}^4$. To find a basis for $\mathcal{N}(A)$, we first find the reduced echelon form for A:

$$A = \begin{bmatrix} 1 & 2 & 0 & 5 \\ 1 & 3 & 1 & 7 \\ 2 & 3 & -1 & 9 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Denoting the unknowns by x_1, x_2, x_3, x_4, then x_1, x_2 and x_4 are the leading variables and x_3 is the free variables. The matrix equation $EX = \mathbf{0}$ is equivalent to the system of equations:

$$\begin{aligned} x_1 - 2x_3 &= 0 \\ x_2 + x_3 &= 0 \\ x_4 &= 0 \end{aligned}$$

Thus, the solutions to $EX = \mathbf{0}$ can be described in vector form as follows:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ -x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

Thus, a basis for $\mathcal{N}(A)$ is:

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

The nullity of A is 1. The rank of A is 3.

PROBLEM 26. To find a basis for $\mathcal{R}(A)$, we use the method of casting out vectors. To do this, we first find the reduced echelon form for A:

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 4 & 2 & 4 \\ 2 & 1 & 5 & -2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & -2 & 4 \\ 0 & -1 & 1 & -2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & -2 & 4 \\ 0 & -1 & 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the leading 1's in E occur in columns 1 and 2, we obtain a basis for $\mathcal{R}(A)$ by choosing those two columns from A. Here is a basis for $\mathcal{R}(A)$:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \right\}$$

Thus, $\dim(\mathcal{R}(A)) = 2$. Since $\dim(\mathcal{R}(A)) = \text{rank}(A)$, we have $\text{rank}(A) = 2$. Therefore, the nullity of A is $\dim(\mathcal{N}(A)) = 4$ - $\text{rank}(A) = 4$ - $2 = 2$.

PROBLEM A: This is question 1 from the Autumn, 2007 sample exam.

(a) We first reduce A to its reduced echelon form E:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The null space of A is defined to be $\mathcal{N}(A) = \{X \ : \ AX = \mathbf{0}_2 \ \}$. Here we are using the notation $\mathbf{0}_2$ for the zero-vector in $\mathbf{R}^2$. The matrix equation $AX = \mathbf{0}_2$ has the same solutions as the matrix equation $EX = \mathbf{0}_2$. We find these solutions by re-introducing the unknowns to get the equations

$$1x_1 + 1x_2 + 0x_3 - 1x_4 = 0, \qquad 1x_3 + 2x_4 = 0$$

Note that x_1 and x_3 are the leading variables, x_2 and x_4 are the free variables. We can describe the solutions in vector form:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1x_2 + 1x_4 \\ 1x_2 \\ -2x_4 \\ 1x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

where x_2 and x_4 are arbitrary. As explained in class, we can then write down the following basis for $\mathcal{N}(A)$:

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

(b) The dimension of $\mathcal{N}(A)$ is the number of vectors in a basis for $\mathcal{N}(A)$. This number is 2. Hence the dimension of $\mathcal{N}(A)$ is 2.

(c) To give an example of a spanning set for $\mathcal{N}(A)$ which is not a basis for $\mathcal{N}(A)$, we need to find a set of vectors in $\mathcal{N}(A)$ which contains a basis for $\mathcal{N}(A)$, but which fails to be linearly independent. We need at least three vectors since $\mathcal{N}(A)$ has dimension 2. We can simply modify the answer to part (a) by including one extra vector from $\mathcal{N}(A)$. We include a third vector, namely the sum of the two specified vectors in the answer to (a). Here is that example:

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

That set is a spanning set for $\mathcal{N}(A)$, but not a basis for $\mathcal{N}(A)$.

(d) We are assuming that $\mathcal{R}(C) = \mathcal{N}(A)$. Now $\mathcal{N}(A)$ is a subspace of $\mathbf{R}^4$ and hence $\mathcal{R}(C)$ will also be a subspace of $\mathbf{R}^4$. Recall that the columns of a matrix C form a spanning set for $\mathcal{R}(C)$. Hence each column of C is contained in $\mathcal{R}(C)$. Since $\mathcal{R}(C)$ is a subspace of $\mathbf{R}^4$, as just stated, it follows that each column of C is contained in $\mathbf{R}^4$. Therefore, C has 4 rows. Since C is $n \times n$, we must have $n = 4$.

PROBLEM B. Let A denote the 2×6 zero matrix. The null space for A is a subspace of $\mathbf{R}^6$. By definition, $\mathcal{N}(A) = \{X : AX = \mathbf{0}\}$. Notice that if X is any vector in $\mathbf{R}^6$, then

$$AX = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, every vector X in $\mathbf{R}^6$ satisfies that equation $AX = \mathbf{0}$. Therefore,

$$\mathcal{N}(A) = \mathbf{R}^6$$

The null space of A is $\mathbf{R}^6$.