SOLUTIONS FOR ASSIGNMENT 6.

SECTION 4.1.

In problems 2, 4, 6, 8, and 10, we will use the fact that the characteristic polynomial $p(t)$ of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $p(t) = t^2 - (a+d)t + ad - bc$.

PROBLEM 2. We have $p(t) = t^2 - t - 2 = (t-2)(t+1)$. The eigenvalues for A are the roots of this polynomial. They are $\lambda = 2$ and $\lambda = -1$.

For $\lambda = 2$, we have

$$A - \lambda I_2 = A - 2I_2 = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}$$

Thus, the eigenspace for $\lambda = 2$ is

$$E_2 = \mathcal{N}(A - 2I_2) = \mathcal{N}\left(\begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} \right) = \mathcal{N}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) = Sp \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

For $\lambda = -1$, we have

$$A - \lambda I_2 = A - (-1)I_2 = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$$

Thus, the eigenspace for $\lambda = -1$ is

$$E_{-1} = \mathcal{N}(A - (-1)I_2) = \mathcal{N}\left(\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} \right) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : \quad 3x + y = 0 \right\} = Sp \left\{ \begin{bmatrix} -1/3 \\ 1 \end{bmatrix} \right\}$$

The eigenvectors for the eigenvalue $\lambda = 2$ are the nonzero scalar multiples of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The eigenvectors for the eigenvalue $\lambda = -1$ are the nonzero scalar multiples of $\begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$.

PROBLEM 4. We have $p(t) = t^2 - 5t + 6 = (t-2)(t-3)$. The eigenvalues for A are the roots of this polynomial. They are $\lambda = 2$ and $\lambda = 3$.

For $\lambda = 2$, we have

$$A - \lambda I_2 = A - 2I_2 = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}$$

Thus, the eigenspace for $\lambda = 2$ is

$$E_2 = \mathcal{N}(A - 2I_2) = \mathcal{N}\left(\begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\right)$$

$$= \mathcal{N}\left(\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} x \\ y \end{bmatrix} : \quad x + 2y = 0\right\} = Sp\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$$

For $\lambda = 3$, we have

$$A - \lambda I_2 = A - 3I_2 = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix}$$

Thus, the eigenspace for $\lambda = 3$ is

$$E_3 = \mathcal{N}(A - 3I_2) = \mathcal{N}\left(\begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$$

$$= \mathcal{N}\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} x \\ y \end{bmatrix} : \quad x + y = 0\right\} = Sp\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$$

The eigenvectors for the eigenvalue $\lambda = 2$ are the nonzero scalar multiples of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. The eigenvectors for the eigenvalue $\lambda = 3$ are the nonzero scalar multiples of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

PROBLEM 6. We have $p(t) = t^2 - 0t - 4 = t^2 - 4 = (t-2)(t+2)$. The eigenvalues for A are the roots of this polynomial. They are $\lambda = 2$ and $\lambda = -2$.

For $\lambda = 2$, we have

$$A - \lambda I_2 = A - 2I_2 = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 5 & -5 \end{bmatrix}$$

Thus, the eigenspace for $\lambda = 2$ is

$$E_2 = \mathcal{N}(A - 2I_2) = \mathcal{N}\left(\begin{bmatrix} 1 & -1 \\ 5 & -5 \end{bmatrix}\right)$$

$$= \mathcal{N}\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} x \\ y \end{bmatrix} : \quad x - y = 0\right\} = Sp\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

For $\lambda = -2$, we have

$$A - \lambda I_2 = A - (-2)I_2 = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 5 & -1 \end{bmatrix}$$

Thus, the eigenspace for $\lambda = -2$ is

$$E_{-2} = \mathcal{N}(A - (-2)I_2) = \mathcal{N}\left(\begin{bmatrix} 5 & -1 \\ 5 & -1 \end{bmatrix} \right) = \mathcal{N}\left(\begin{bmatrix} 1 & -1/5 \\ 5 & -1 \end{bmatrix} \right)$$

$$= \mathcal{N}\left(\begin{bmatrix} 1 & -1/5 \\ 0 & 0 \end{bmatrix} \right) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \quad : \quad x - (1/5)y = 0 \right\} = Sp\left\{ \begin{bmatrix} 1/5 \\ 1 \end{bmatrix} \right\}$$

The eigenvectors for the eigenvalue $\lambda = 2$ are the nonzero scalar multiples of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The eigenvectors for the eigenvalue $\lambda = -2$ are the nonzero scalar multiples of $\begin{bmatrix} 1/5 \\ 1 \end{bmatrix}$.

PROBLEM 8. We have $p(t) = t^2 - 4t + 4 = (t-2)(t-2) = (t-2)^2$. The eigenvalues for A are the roots of this polynomial. There is just one eigenvalue: They are $\lambda = 2$.

We have

$$A - \lambda I_2 = A - 2I_2 = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$$

Thus, the eigenspace for $\lambda = 2$ is

$$E_2 = \mathcal{N}(A - 2I_2) = \mathcal{N}\left(\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \right) = \mathcal{N}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) = Sp\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

Thus, the eigenvectors for A for the eigenvalue $\lambda = 2$ are the nonzero scalar multiples of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

PROBLEM 10. We have $p(t) = t^2 - 9t + 0 = (t-9)t = (t-9)(t-0)$. The eigenvalues for A are the roots of this polynomial. They are $\lambda = 9$ and $\lambda = 0$.

For $\lambda = 9$, we have

$$A - \lambda I_2 = A - 9I_2 = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} -8 & 2 \\ 4 & -1 \end{bmatrix}$$

Thus, the eigenspace for $\lambda = 9$ is

$$E_9 = \mathcal{N}(A - 9I_2) = \mathcal{N}\left(\begin{bmatrix} -8 & 2 \\ 4 & -1 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} 1 & -1/4 \\ 4 & -1 \end{bmatrix}\right)$$

$$= \mathcal{N}\left(\begin{bmatrix} 1 & -1/4 \\ 0 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} x \\ y \end{bmatrix} \quad : \quad x - (1/4)y = 0\right\} = Sp\left\{\begin{bmatrix} 1/4 \\ 1 \end{bmatrix}\right\}$$

For $\lambda = 0$, we have

$$A - \lambda I_2 = A - 0I_2 = A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$$

Thus, the eigenspace for $\lambda = 0$ is

$$E_0 = \mathcal{N}(A) = \mathcal{N}\left(\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} x \\ y \end{bmatrix} \quad : \quad x + 2y = 0\right\} = Sp\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$$

Thus, the eigenvectors for A for the eigenvalue $\lambda = 9$ are the nonzero scalar multiples of $\begin{bmatrix} 1/4 \\ 1 \end{bmatrix}$. The eigenvectors for A for the eigenvalue $\lambda = 0$ are the nonzero scalar multiples of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

SECTION 4.2

PROBLEM 10. $Det\left(\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}\right) = 2 \cdot 6 - 3 \cdot 4 = 12 - 12 = 0$. The matrix $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ is singular.

PROBLEM 12. $Det\left(\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 4 & 2 & 10 \end{bmatrix}\right) = 1Det\left(\begin{bmatrix} 3 & 7 \\ 2 & 10 \end{bmatrix}\right) - 2Det\left(\begin{bmatrix} 2 & 7 \\ 4 & 10 \end{bmatrix}\right) + 4Det\left(\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}\right)$

$= 1(3 \cdot 10 - 7 \cdot 2) - 2(2 \cdot 10 - 7 \cdot 4) + 4(2 \cdot 2 - 3 \cdot 4) = 1 \cdot 16 - 2 \cdot (-8) + 4 \cdot (-8) = 0$

The matrix $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 4 & 2 & 10 \end{bmatrix}$ is singular.

PROBLEM 14. $Det\left(\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ -1 & 1 & 1 \end{bmatrix}\right) = 1Det\left(\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\right) - 2Det\left(\begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}\right) + 1Det\left(\begin{bmatrix} 0 & 3 \\ -1 & 1 \end{bmatrix}\right)$

$= 1(3 \cdot 1 - 2 \cdot 1) - 2(0 \cdot 1 - 2 \cdot (-1)) + 1(0 \cdot 1 - 3 \cdot (-1)) = 1 \cdot 1 - 2 \cdot 2 + 1 \cdot 3 = 0.$

The matrix $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ -1 & 1 & 1 \end{bmatrix}$ is singular.

PROBLEM 28. It is given that $Det(A) = 3$ and $Det(B) = 5$. Using the multiplicative property of the determinant, we have:

$Det(A^2B) = Det(AAB) = Det(A)Det(A)Det(B) = 3 \cdot 3 \cdot 5 = 45.$

PROBLEM 30. It is given that $Det(A) = 3$ and $Det(B) = 5$. Using the multiplicative property of the determinant, we have $Det(A^{-1}) = 1/3$ and $Det(B^{-1}) = 1/5$ and

$Det(AB^{-1}A^{-1}B) = Det(A)Det(B^{-1})Det(A^{-1})Det(B) = 3 \cdot (1/5) \cdot (1/3) \cdot 5 = 1.$

SECTION 4.4.

PROBLEM 4. We will use the fact that the characteristic polynomial $p(t)$ of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $p(t) = t^2 - (a+d)t + ad - bc$. The characteristic polynomial for the matrix $\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$ is $p(t) = t^2 - 1t - 2 = (t-2)(t+1)$.

The eigenvalues for the matrix are the roots of this polynomial. They are $\lambda = 2$ and $\lambda = -1$. The algebraic multiplicity is 1 for each of the eigenvalues.

PROBLEM 6. The characteristic polynomial for the matrix $\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$ is

$$p(t) = t^2 - 5t + 0 = (t-5)t = (t-5)(t-0)$$

The eigenvalues for the matrix are the roots of this polynomial. They are $\lambda = 5$ and $\lambda = 0$. The algebraic multiplicity is 1 for each of the eigenvalues.

PROBLEM 8. Let A denote the matrix in this problem. Then

$$A - tI_3 = \begin{bmatrix} -2 & -1 & 0 \\ 0 & 1 & 1 \\ -2 & -2 & -1 \end{bmatrix} - t \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 \\ 0 & 1 & 1 \\ -2 & -2 & -1 \end{bmatrix} - \begin{bmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{bmatrix}$$

$$= \begin{bmatrix} -2-t & -1 & 0 \\ 0 & 1-t & 1 \\ -2 & -2 & -1-t \end{bmatrix}$$

The characteristic polynomial $p(t)$ is the determinant of this matrix. We will calculate it by expanding along row 1. We have

$$p(t) = det\left(\begin{bmatrix} -2-t & -1 & 0 \\ 0 & 1-t & 1 \\ -2 & -2 & -1-t \end{bmatrix}\right)$$

$$= (-2-t)det\left(\begin{bmatrix} 1-t & 1 \\ -2 & -1-t \end{bmatrix}\right) - (-1)det\left(\begin{bmatrix} 0 & 1 \\ -2 & -1-t \end{bmatrix}\right) + 0det\left(\begin{bmatrix} 0 & 1-t \\ -2 & -2 \end{bmatrix}\right)$$
$$= (-2-t)\Big((1-t)(-1-t) - 1(-2)\Big) - (-1)\Big(0(-1-t) - 1(-2)\Big) + 0\Big(0(-2) - (-2)(1-t)\Big)$$
$$= (-2-t)\Big(-(1-t^2) + 2\Big) + 1(2) = (-2-t)(t^2+1) + 2 = -2t^2 - t^3 - 2 - t + 2$$
$$= -t^3 - 2t^2 - t = -t(t^2 + 2t + 1) = -t(t+1)(t+1) = -t(t+1)^2$$

The eigenvalues of the matrix A are $\lambda = 0$ and $\lambda = -1$. The eigenvalue $\lambda = 0$ has algebraic multiplicity 1 and the eigenvalue $\lambda = -1$ has algebraic multiplicity 2.

SECTION 4.5.

PROBLEM 2. We have $A - \lambda I_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. We are asked to find a basis for $E_1 = \mathcal{N}\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)$. We have

$$\mathcal{N}\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} x \\ y \end{bmatrix} : \quad x - y = 0\right\} = Sp\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

The set $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ is a basis for the eigenspace E_1. The algebraic multiplicity of the eigenvalue $\lambda = 1$ is 1. The geometric multiplicity of this eigenvalue for A is $\dim(E_1)$ which is equal to 1.

PROBLEM 4. We have $C - \lambda I_3 = \begin{bmatrix} -6 & -1 & 2 \\ 3 & 2 & 0 \\ -14 & -2 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -7 & -1 & 2 \\ 3 & 1 & 0 \\ -14 & -2 & 4 \end{bmatrix}$.

The eigenspace E_1 is the null space for this matrix. We have

$$E_1 = \mathcal{N}\left(\begin{bmatrix} -7 & -1 & 2 \\ 3 & 1 & 0 \\ -14 & -2 & 4 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} 1 & 1/7 & -2/7 \\ 3 & 1 & 0 \\ -14 & -2 & 4 \end{bmatrix}\right)$$

$$= \mathcal{N}\left(\begin{bmatrix} 1 & 1/7 & -2/7 \\ 0 & 4/7 & 6/7 \\ 0 & 0 & 0 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} 1 & 1/7 & -2/7 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \quad x - (1/2)z = 0, \quad y + (3/2)z = 0 \right\} = Sp\left\{ \begin{bmatrix} 1/2 \\ -3/2 \\ 1 \end{bmatrix} \right\}$$

The set $\left\{ \begin{bmatrix} 1/2 \\ -3/2 \\ 1 \end{bmatrix} \right\}$ is a basis for E_1. The algebraic multiplicity of the eigenvalue $\lambda = 1$ is 2. The geometric multiplicity of this eigenvalue for A is $\dim(E_1)$ which is equal to 1.

PROBLEM 6. We have $D - \lambda I_3 = \begin{bmatrix} -7 & 4 & -3 \\ 8 & -3 & 3 \\ 32 & -16 & 13 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -8 & 4 & -3 \\ 8 & -4 & 3 \\ 32 & -16 & 12 \end{bmatrix}$.

The eigenspace E_1 is the null space for this matrix. We have

$$E_1 = \mathcal{N}\left(\begin{bmatrix} -8 & 4 & -3 \\ 8 & -4 & 3 \\ 32 & -16 & 12 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} -8 & 4 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$= \mathcal{N}\left(\begin{bmatrix} 1 & -1/2 & 3/8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \quad x - (1/2)z + (3/8)z = 0 \right\} = \left\{ \begin{bmatrix} (1/2)y - (3/8)z \\ y \\ z \end{bmatrix} \right\}$$

$$= \left\{ y \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -3/8 \\ 0 \\ 1 \end{bmatrix} \right\} = Sp\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -3/8 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The set

$$\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -3/8 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for E_1. The algebraic multiplicity of the eigenvalue $\lambda = 1$ is 3. The geometric multiplicity of this eigenvalue for A is $\dim(E_1)$ which is equal to 2.

A. This question concerns the matrix $S = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$. We first consider the eigenvalue $\lambda = 1$. The eigenspace E_1 is the null space for the matrix

$$S - I_3 = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & -2/3 \\ 2/3 & -2/3 & -2/3 \end{bmatrix}$$

We can find a basis for the null space of $S - I_3$ by row-reduction:

$$\begin{bmatrix} -2/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & -2/3 \\ 2/3 & -2/3 & -2/3 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & -1 \\ 2/3 & -2/3 & -2/3 \\ 2/3 & -2/3 & -2/3 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus,

$$E_1 = \mathcal{N}(S - I_3) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \quad x - y - z = 0 \right\} = \left\{ \begin{bmatrix} y + z \\ y \\ z \end{bmatrix} \right\}$$

$$= \left\{ y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} = Sp \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The set

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for the eigenspace E_1.

Now we consider the eigenspace E_{-1}. The eigenspace E_{-1} is the null space for the matrix

$$S - (-1)I_3 = S + I_3 = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4/3 & 2/3 & 2/3 \\ 2/3 & 4/3 & -2/3 \\ 2/3 & -2/3 & 4/3 \end{bmatrix}$$

We can find a basis for the null space of $S + I_3$ by row-reduction:

$$\begin{bmatrix} 4/3 & 2/3 & 2/3 \\ 2/3 & 4/3 & -2/3 \\ 2/3 & -2/3 & 4/3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1/2 & 1/2 \\ 2/3 & 4/3 & -2/3 \\ 2/3 & -2/3 & 4/3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus,

$$E_{-1} = \mathcal{N}(S + I_3) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \quad x + z = 0, \quad y - z = 0 \right\} = \left\{ \begin{bmatrix} -z \\ z \\ z \end{bmatrix} \right\} = Sp \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

The set $\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for E_{-1}.