

# PARTIAL DATA FOR THE CALDERÓN PROBLEM IN TWO DIMENSIONS

OLEG YU. IMANUVILOV, GUNTHER UHLMANN, AND MASAHIRO YAMAMOTO

ABSTRACT. For the conductivity equation in a simply connected two-dimensional bounded domain, we show that Dirichlet data on an open subset of the boundary and Neumann data on slightly larger set than the complement, uniquely determine the conductivity. We prove also a similar result for the Schrödinger equation. Using Carleman estimates with degenerate weights we construct appropriate complex geometrical optics solutions to prove the results.

## 1. Introduction

This paper is concerned with the Electrical Impedance Tomography (EIT) inverse problem. The EIT inverse problem consists in determining the electrical conductivity of a body by making voltage and current measurements at the boundary of the body. Substantial progress has been made on this problem since Calderón's pioneer contribution [8]. This inverse problem is known also as the Calderón problem. This problem can be reduced to studying the Dirichlet-to-Neumann (DN) map associated to the Schrödinger equation. A key ingredient in several of the results is the construction of complex geometrical optics solutions for the Schrödinger equation (see [24] for a recent survey). By this method in dimensions  $n \geq 3$  for the conductivity equation, the first global uniqueness result for  $C^2$  conductivities was proven in [22] and the regularity was improved to having  $3/2$  derivatives in [4] and [20]. More singular conormal conductivities were considered in [12]. The uniqueness results were proven also for the Schrödinger equation.

In two dimensions the first global uniqueness result for the Calderón problem with full data is in [19] for  $C^2$ -conductivities, and this was improved to Lipschitz conductivities in [5] and for merely  $L^\infty$  conductivities in [2]. However, the corresponding result for the Schrödinger equation was not known until the recent breakthrough [6]. As for the uniqueness in determining two coefficients, see [9]. In [15] it is shown in two dimensions that one can uniquely determine the magnetic field and the electrical potential from the DN map associated to the Pauli Hamiltonian.

If the DN map is measured only on a part of the boundary, then much less is known. We only review here the results where no a-priori information is assumed. In dimensions  $n \geq 3$  a global result is shown in [7] where partial measurements of the DN map are assumed: More precisely, for  $C^2$  conductivities if we measure the DN map restricted to a slightly larger than the half of the boundary, then one can determine uniquely the potential. The proof relies

---

First author partly supported by NSF grant DMS 0808130.

Second author partly supported by NSF and a Walker Family Endowed Professorship.

on a Carleman estimate with a linear weight function. The Carleman estimate can also be used to construct complex geometrical optics solutions for the Schrödinger equation. In [17] the regularity assumption on the conductivity was relaxed to  $C^{3/2+\ell}$  with some  $\ell > 0$ . Stability estimates for the uniqueness result of [7] were given in [13]. Stability estimates for the magnetic Schrödinger operator with partial data in the setting of [7] can be found in [23].

In [16], the result in [7] was generalized and it is shown that by all possible pairs of Dirichlet data on an arbitrary open subset  $\Gamma_+$  of the boundary and Neumann data on a slightly larger open subset than  $\partial\Omega \setminus \Gamma_+$ , one can uniquely determine the potential. The case of the magnetic Schrödinger equation was considered in [10] and improvement on the regularity of the coefficients can be found in [18].

In this paper we show a result similar to [16] in two dimensions by constructing complex geometrical optics solutions with degenerate weights. We note that in two dimensions the problem is formally determined while in three or higher dimensions it is overdetermined. We now state the main result more precisely.

Let  $\Omega \subset \mathbf{R}^2$  be a simply connected bounded domain with smooth boundary. The electrical conductivity of  $\Omega$  is represented by a bounded and positive function  $\gamma(x)$ . In the absence of sinks or sources of current, the potential  $u \in H^1(\Omega)$  with given boundary voltage potential  $f \in H^{\frac{1}{2}}(\partial\Omega)$  is a solution of the Dirichlet boundary value problem

$$(1.1) \quad \begin{aligned} \operatorname{div}(\gamma \nabla u) &= 0 \text{ in } \Omega, \\ u|_{\partial\Omega} &= f. \end{aligned}$$

The Dirichlet to Neumann (DN) map, or voltage to current map, is given by

$$(1.2) \quad \Lambda_\gamma(f) = \gamma \frac{\partial u}{\partial \nu} \Big|_{\partial\Omega},$$

where  $\nu$  denotes the unit outer normal to  $\partial\Omega$ . This problem can be reduced to studying the set of Cauchy data for the Schrödinger equation with the potential  $q$  given by:

$$(1.3) \quad q = \frac{\Delta \sqrt{\gamma}}{\sqrt{\gamma}}.$$

$$(1.4) \quad \tilde{C}_q = \left\{ \left( u|_{\partial\Omega}, \frac{\partial u}{\partial \nu} \Big|_{\partial\Omega} \right) \mid (\Delta + q)u = 0 \text{ on } \Omega, u \in H^1(\Omega) \right\}.$$

We have  $\tilde{C}_q \subset H^{\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ .

By using a conformal map, thanks to the Kellogg-Warchawski theorem (see e.g. p. 42 [21]), without loss of generality we assume that  $\Omega = \{x \in \mathbf{R}^2 \mid |x| < 1\}$ .

Let  $\Gamma_- = \{(\cos \theta, \sin \theta) \mid \theta \in (-\theta_0, \theta_0)\}$  be a connected subdomain in  $\partial\Omega$  and  $\theta_0 \in (0, \pi)$ ,  $\hat{x}_\pm$  the boundary of  $\Gamma_-$ :  $\partial\Gamma_- = \{\hat{x}_\pm\}$ . Denote  $\Gamma_+ = S^1 \setminus \Gamma_-$ . Let  $\epsilon > 0$  be a small number such that  $\theta_0 + \epsilon \in (0, \pi]$ . Denote by  $\Gamma_{-, \epsilon} = \{(\cos \theta, \sin \theta) \mid \theta \in (-\theta_0 - \epsilon, \theta_0 + \epsilon)\}$  and by  $\hat{x}_{\pm, \epsilon}$  the endpoints of  $\Gamma_{-, \epsilon}$ .

We have

**Theorem 1.1.** *Let  $q_j \in C^{1+\ell}(\overline{\Omega})$ ,  $j = 1, 2$  for some positive  $\ell$ . Consider the following sets of partial Cauchy data:*

$$(1.5) \quad \mathcal{C}_{q_j, \epsilon} = \left\{ \left( u|_{\Gamma_+}, \frac{\partial u}{\partial \nu} \Big|_{\Gamma_{-, \epsilon}} \right) \mid (\Delta + q_j)u = 0 \text{ in } \Omega, u|_{\Gamma_-} = 0, u \in H^1(\Omega) \right\}, \quad j = 1, 2.$$

Assume

$$\mathcal{C}_{q_1, \epsilon} = \mathcal{C}_{q_2, \epsilon}$$

with some  $\epsilon > 0$ . Then

$$q_1 = q_2.$$

As a direct consequence of Theorem 1.1 we have

**Corollary 1.1.** *Let  $\gamma_1, \gamma_2$  be strictly positive functions and there exists some positive number  $\ell$  such that  $\gamma_1, \gamma_2 \in C^{3+\ell}(\overline{\Omega})$ . Assume that  $\gamma_1 = \gamma_2$  on  $\partial\Omega$  and*

$$\gamma_1 \frac{\partial u}{\partial \nu} = \gamma_2 \frac{\partial u}{\partial \nu} \quad \text{on } \Gamma_{-, \epsilon} \text{ for all } u \in H^{\frac{1}{2}}(\partial\Omega), \text{ supp } u \subset \Gamma_+.$$

Then  $\gamma_1 = \gamma_2$ .

The proof of Theorem 1.1 uses Carleman estimates for the Laplacian with degenerate limiting Carleman weights. The results of [7] and [16] use complex geometrical optics solutions of the form

$$(1.6) \quad u = e^{\tau(\varphi + \sqrt{-1}\psi)}(a + r),$$

where  $\nabla\varphi \cdot \nabla\psi = 0$ ,  $|\nabla\varphi|^2 = |\nabla\psi|^2$  and  $\varphi$  is a limiting Carleman weight and  $a$  is smooth and non-vanishing and  $\|r\|_{L^2(\Omega)} = O(\frac{1}{\tau})$ ,  $\|r\|_{H^1(\Omega)} = O(1)$ . Examples of limiting Carleman weights are the linear phase  $\varphi(x) = x \cdot \omega$  with  $\omega \in S^{n-1}$  which was used in [7], and the non-linear phase  $\varphi(x) = \ln|x - x_0|$ , where  $x_0 \in \mathbf{R}^n \setminus \overline{\Omega}$  which was used in [16]. For a complete characterization of possible local Carleman weights in the Euclidean space and more general manifolds, see [11].

In two dimensions the limiting Carleman weights are harmonic functions so that there is a larger class of complex geometrical optics solutions. This freedom was used in [25] to determine inclusions for a large class of systems in two dimensions. In particular, one can use the harmonic function  $\varphi = \text{Re } z^n$  as limiting Carleman weight, assuming that 0 is outside the domain.

In this paper we construct complex geometrical optics solutions of the form

$$(1.7) \quad u = e^{\tau(\varphi + \sqrt{-1}\psi)}(a + r) + u_r,$$

where  $u_r$  is a ‘‘reflected’’ term to guarantee that the solution vanishes in particular subsets of the boundary,  $\varphi$  is a harmonic function having a finite number of non-degenerate critical points in  $\Omega$ , and  $\psi$  is the corresponding conjugate harmonic function. However we need to modify the form with  $\varphi$  harmonic but having non-degenerate critical points. Solutions as in (1.6) with degenerate harmonic functions were also used in [6] but here the phase function needs to satisfy further restrictions in order to use them for the partial data problem. Another complication is that the correction term  $r$  and the reflected term  $u_r$  do not have the same

asymptotic behavior in  $\tau$  as in [16] because of the degeneration of the phase, so that one needs to further decompose these terms and analyze their asymptotic behavior in  $\tau$ . See section 3 for more details. In section 2 we prove a general Carleman estimate with degenerate weights. Finally in section 4 we prove Theorem 1.1.

## 2. Carleman estimates with degenerate weights

Throughout the paper we use the following notations:

**Notations**  $i = \sqrt{-1}$ ,  $x_1, x_2, \xi_1, \xi_2 \in \mathbf{R}$ ,  $z = x_1 + ix_2$ ,  $\zeta = \xi_1 + i\xi_2$ ,  $\frac{\partial}{\partial z} = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$ ,  $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$ ,  $H^{1,\tau}(\Omega)$  denotes the space  $H^1(\Omega)$  with norm  $\|v\|_{H^{1,\tau}(\Omega)}^2 = \|v\|_{H^1(\Omega)}^2 + \tau^2\|v\|_{L^2(\Omega)}^2$ . The tangential derivative on the boundary is given by  $\partial_\tau = \nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2}$ , with  $\nu = (\nu_1, \nu_2)$  the unit outer normal to  $\partial\Omega$ ,  $B(\hat{x}, \delta) = \{x \in \mathbf{R}^2 \mid |x - \hat{x}| < \delta\}$ ,  $S^1 = \{x \in \mathbf{R}^2 \mid |x| = 1\}$ ,  $f(x) : \mathbf{R}^2 \rightarrow \mathbf{R}^1$ ,  $f''$  is the Hessian matrix with entries  $\frac{\partial^2 f}{\partial x_i \partial x_j}$ .

Let  $\Phi(z) = \varphi_1(x_1, x_2) + i\varphi_2(x_1, x_2)$  be a holomorphic function in a domain  $\Omega_0$ , given that  $\bar{\Omega} \subset \Omega_0$ ,

$$(2.1) \quad \frac{\partial \Phi(z)}{\partial \bar{z}} = 0 \quad \text{in } \Omega_0, \quad \Phi \in C^2(\bar{\Omega}_0).$$

Denote by  $\mathcal{H}$  the set of critical points of a function  $\Phi$

$$\mathcal{H} = \left\{ z \in \bar{\Omega} \mid \frac{\partial \Phi}{\partial z}(z) = 0 \right\}.$$

Assume that  $\Phi$  has no critical points at the boundary and nondegenerate critical points in the interior;

$$(2.2) \quad \mathcal{H} \cap \partial\Omega = \{\emptyset\}, \quad \Phi''(z) \neq 0 \quad \forall z \in \mathcal{H}.$$

Then  $\Phi$  we have only a finite number of critical points:

$$(2.3) \quad \text{card } \mathcal{H} < \infty.$$

Denote  $\frac{\partial \Phi}{\partial z}(z) = \psi_1(x_1, x_2) + i\psi_2(x_1, x_2)$ .

We will prove Carleman estimates for the conjugated operator

$$\Delta_\tau = e^{\tau\varphi_1} \Delta e^{-\tau\varphi_1}.$$

We will use the factorization

$$(2.4) \quad \Delta_\tau \tilde{v} = \left( 2 \frac{\partial}{\partial z} - \tau \frac{\partial \Phi}{\partial z} \right) \left( 2 \frac{\partial}{\partial \bar{z}} - \tau \frac{\partial \bar{\Phi}}{\partial \bar{z}} \right) \tilde{v} = \left( 2 \frac{\partial}{\partial \bar{z}} - \tau \frac{\partial \bar{\Phi}}{\partial \bar{z}} \right) \left( 2 \frac{\partial}{\partial z} - \tau \frac{\partial \Phi}{\partial z} \right) \tilde{v}$$

and prove Carleman estimates first for every term in the factorization.

**Proposition 2.1.** *Let  $\Phi$  satisfy (2.1) and (2.2). Let  $\tilde{f} \in L^2(\Omega)$ , and  $\tilde{v}$  be solution to the problem*

$$(2.5) \quad 2 \frac{\partial \tilde{v}}{\partial z} - \tau \frac{\partial \Phi}{\partial z} \tilde{v} = \tilde{f} \quad \text{in } \Omega$$

or  $\tilde{v}$  be solution to the problem

$$(2.6) \quad 2\frac{\partial\tilde{v}}{\partial\bar{z}} - \tau\frac{\partial\bar{\Phi}}{\partial\bar{z}}\tilde{v} = \tilde{f} \quad \text{in } \Omega.$$

In the case (2.5) we have

$$(2.7) \quad \left\| \left( \frac{\partial}{\partial x_1} - i\psi_2\tau \right) \tilde{v} \right\|_{L^2(\Omega)}^2 - \tau \int_{\partial\Omega} (\nabla\varphi_1, \nu) |\tilde{v}|^2 d\sigma \\ + \operatorname{Re} \int_{\partial\Omega} i \left( \left( \nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2} \right) \tilde{v} \right) \bar{\tilde{v}} d\sigma + \left\| \left( i \frac{\partial}{\partial x_2} + \tau\psi_1 \right) \tilde{v} \right\|_{L^2(\Omega)}^2 = \|\tilde{f}\|_{L^2(\Omega)}^2,$$

while in the case (2.6) we have

$$(2.8) \quad \left\| \left( \frac{\partial}{\partial x_1} + i\psi_2\tau \right) \tilde{v} \right\|_{L^2(\Omega)}^2 - \tau \int_{\partial\Omega} (\nabla\varphi_1, \nu) |\tilde{v}|^2 d\sigma + \operatorname{Re} \int_{\partial\Omega} i \left( \left( -\nu_2 \frac{\partial}{\partial x_1} + \nu_1 \frac{\partial}{\partial x_2} \right) \tilde{v} \right) \bar{\tilde{v}} d\sigma \\ + \left\| \left( i \frac{\partial}{\partial x_2} - \psi_1\tau \right) \tilde{v} \right\|_{L^2(\Omega)}^2 = \|\tilde{f}\|_{L^2(\Omega)}^2.$$

*Proof.* We prove the statement of the proposition first for the equation  $2\frac{\partial\tilde{v}}{\partial\bar{z}} - \tau\frac{\partial\bar{\Phi}}{\partial\bar{z}}\tilde{v} = \tilde{f}$ . Since  $2\frac{\partial}{\partial\bar{z}} - \tau\frac{\partial\bar{\Phi}}{\partial\bar{z}} = \left( \frac{\partial}{\partial x_1} - i\psi_2\tau \right) + \left( i\frac{\partial}{\partial x_2} - \psi_1\tau \right)$ , taking the  $L^2$ - norms of the right and left hand sides of (2.5) we have

$$\left\| \left( \frac{\partial}{\partial x_1} - i\psi_2\tau \right) \tilde{v} \right\|_{L^2(\Omega)}^2 + 2\operatorname{Re} \left( \left( \frac{\partial}{\partial x_1} - i\psi_2\tau \right) \tilde{v}, \left( -i\frac{\partial}{\partial x_2} - \psi_1\tau \right) \tilde{v} \right)_{L^2(\Omega)} \\ + \left\| \left( -i\frac{\partial}{\partial x_2} - \psi_1\tau \right) \tilde{v} \right\|_{L^2(\Omega)}^2 = \|\tilde{f}\|_{L^2(\Omega)}^2.$$

Since we take the commutator to have  $\left[ \left( \frac{\partial}{\partial x_1} - i\psi_2\tau \right), \left( i\frac{\partial}{\partial x_2} - \psi_1\tau \right) \right] \equiv 0$ , we obtain

$$\left\| \left( \frac{\partial}{\partial x_1} - i\psi_2\tau \right) \tilde{v} \right\|_{L^2(\Omega)}^2 + \left( \left( \frac{\partial}{\partial x_1} - i\psi_2\tau \right) \tilde{v}, \overline{(-i\nu_2\tilde{v})} \right)_{L^2(\partial\Omega)} + \left( \overline{\nu_1\tilde{v}}, \left( -i\frac{\partial}{\partial x_2} - \psi_1\tau \right) \tilde{v} \right)_{L^2(\partial\Omega)} \\ + \left\| \left( i\frac{\partial}{\partial x_2} + \psi_1\tau \right) \tilde{v} \right\|_{L^2(\Omega)}^2 = \|\tilde{f}\|_{L^2(\Omega)}^2.$$

This equality implies

$$\left\| \left( \frac{\partial}{\partial x_1} - i\psi_2\tau \right) \tilde{v} \right\|_{L^2(\Omega)}^2 - \tau \int_{\partial\Omega} (\psi_1\nu_1 - \psi_2\nu_2) |\tilde{v}|^2 d\sigma + \int_{\partial\Omega} i \left( \left( \nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2} \right) \tilde{v} \right) \bar{\tilde{v}} d\sigma \\ + \left\| \left( i\frac{\partial}{\partial x_2} + \psi_1\tau \right) \tilde{v} \right\|_{L^2(\Omega)}^2 = \|\tilde{f}\|_{L^2(\Omega)}^2.$$

Finally by (2.1) we observe that  $\psi_1 = \frac{1}{2} \left( \frac{\partial\varphi_1}{\partial x_1} + \frac{\partial\varphi_2}{\partial x_2} \right) = \frac{\partial\varphi_1}{\partial x_1}$  and  $\psi_2 = \frac{1}{2} \left( \frac{\partial\varphi_2}{\partial x_1} - \frac{\partial\varphi_1}{\partial x_2} \right) = -\frac{\partial\varphi_1}{\partial x_2}$ . Therefore from the above equality, (2.7) follows immediately.

Now we prove the statement of the theorem for the equation (2.6). Since  $2\frac{\partial}{\partial \bar{z}} - \tau\frac{\partial \bar{\Phi}}{\partial \bar{z}} = \left(\frac{\partial}{\partial x_1} + i\psi_2\tau\right) + \left(-\frac{\partial}{i\partial x_2} - \psi_1\tau\right)$ , taking the  $L^2$ - norms of the right and left hand sides of (2.6) we have

$$\begin{aligned} \left\| \left( \frac{\partial}{\partial x_1} + i\psi_2\tau \right) \tilde{v} \right\|_{L^2(\Omega)}^2 + 2\operatorname{Re} \left( \left( \frac{\partial}{\partial x_1} + i\psi_2\tau \right) \tilde{v}, \left( i\frac{\partial}{\partial x_2} - \psi_1\tau \right) \tilde{v} \right)_{L^2(\Omega)} \\ + \left\| \left( i\frac{\partial}{\partial x_2} - \psi_1\tau \right) \tilde{v} \right\|_{L^2(\Omega)}^2 = \|\tilde{f}\|_{L^2(\Omega)}^2. \end{aligned}$$

Since  $\left[\left(\frac{\partial}{\partial x_1} + i\psi_2\tau\right), \left(\frac{\partial}{i\partial x_2} + \psi_1\tau\right)\right] \equiv 0$ , we obtain

$$\begin{aligned} \left\| \left( \frac{\partial}{\partial x_1} + i\psi_2\tau \right) \tilde{v} \right\|_{L^2(\Omega)}^2 + \left( \left( \frac{\partial}{\partial x_1} + i\psi_2\tau \right) \tilde{v}, \overline{(i\nu_2\tilde{v})} \right)_{L^2(\partial\Omega)} + \left( \overline{\nu_1\tilde{v}}, \left( i\frac{\partial}{\partial x_2} - \psi_1\tau \right) \tilde{v} \right)_{L^2(\partial\Omega)} \\ + \left\| \left( i\frac{\partial}{\partial x_2} - \psi_1\tau \right) \tilde{v} \right\|_{L^2(\Omega)}^2 = \|\tilde{f}\|_{L^2(\Omega)}^2. \end{aligned}$$

This equality implies

$$\begin{aligned} \left\| \left( \frac{\partial}{\partial x_1} + i\psi_2\tau \right) \tilde{v} \right\|_{L^2(\Omega)}^2 - \tau \int_{\partial\Omega} (\psi_1\nu_1 - \psi_2\nu_2) |\tilde{v}|^2 d\sigma + \int_{\partial\Omega} i \left( \left( -\nu_2\frac{\partial}{\partial x_1} + \nu_1\frac{\partial}{\partial x_2} \right) \tilde{v} \right) \overline{\tilde{v}} d\sigma \\ + \left\| \left( i\frac{\partial}{\partial x_2} - \psi_1\tau \right) \tilde{v} \right\|_{L^2(\Omega)}^2 = \|\tilde{f}\|_{L^2(\Omega)}^2. \end{aligned}$$

Finally we observe that  $\psi_1 = \frac{1}{2}\left(\frac{\partial\varphi_1}{\partial x_1} + \frac{\partial\varphi_2}{\partial x_2}\right) = \frac{\partial\varphi_1}{\partial x_1}$  and  $\psi_2 = \frac{1}{2}\left(\frac{\partial\varphi_2}{\partial x_1} - \frac{\partial\varphi_1}{\partial x_2}\right) = -\frac{\partial\varphi_1}{\partial x_2}$ . Thus estimate (2.8) follows immediately from the above equality, finishing the proof of the proposition.  $\square$

Let  $u$  solve the boundary value problem

$$(2.9) \quad \Delta u = f \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0.$$

Denote

$$\partial\Omega_+ = \{(x_1, x_2) \in \partial\Omega | (\nabla\varphi_1, \nu) > 0\}$$

and

$$\partial\Omega_- = \{(x_1, x_2) \in \partial\Omega | (\nabla\varphi_1, \nu) < 0\}.$$

The main result of this section is the following Carleman estimate with degenerate weights.

**Theorem 2.1.** *Suppose that  $\Phi$  satisfies (2.1) and (2.2). Let  $f \in L^2(\Omega)$ , and let  $u$  be a solution to (2.9) with  $u \in H^1(\Omega)$ . Then there exist positive constants  $C > 0$  and  $\tau_0$  such that*

for all  $\tau \geq \tau_0$ :

$$(2.10) \quad \begin{aligned} & \tau \|ue^{\tau\varphi_1}\|_{L^2(\Omega)}^2 + \|ue^{\tau\varphi_1}\|_{H^1(\Omega)}^2 + \tau^2 \left\| \left| \frac{\partial\Phi}{\partial z} \right| ue^{\tau\varphi_1} \right\|_{L^2(\Omega)}^2 - \tau \int_{\partial\Omega_-} (\nu, \nabla\varphi_1) \left| \frac{\partial u}{\partial\nu} \right|^2 e^{2\tau\varphi_1} d\sigma \\ & \leq C \left( \|fe^{\tau\varphi_1}\|_{L^2(\Omega)}^2 + \tau \int_{\partial\Omega_+} (\nu, \nabla\varphi_1) \left| \frac{\partial u}{\partial\nu} \right|^2 e^{2\tau\varphi_1} d\sigma \right). \end{aligned}$$

*Proof.* As indicated earlier we can take  $\Omega$  to be the unit ball. Denote  $\tilde{v} = ue^{\tau\varphi_1}$ . Without the loss of generality we may assume that  $u$  is a real valued function. By (2.4)

$$\Delta_\tau \tilde{v} = \left( 2\frac{\partial}{\partial z} - \tau\frac{\partial\Phi}{\partial z} \right) \left( 2\frac{\partial}{\partial\bar{z}} - \tau\frac{\partial\bar{\Phi}}{\partial\bar{z}} \right) \tilde{v} = \left( 2\frac{\partial}{\partial\bar{z}} - \tau\frac{\partial\bar{\Phi}}{\partial\bar{z}} \right) \left( 2\frac{\partial}{\partial z} - \tau\frac{\partial\Phi}{\partial z} \right) \tilde{v} = fe^{\tau\varphi_1}.$$

Denote  $\tilde{w}_1 = (2\frac{\partial}{\partial\bar{z}} - \tau\frac{\partial\bar{\Phi}}{\partial\bar{z}})\tilde{v}$ ,  $\tilde{w}_2 = (2\frac{\partial}{\partial z} - \tau\frac{\partial\Phi}{\partial z})\tilde{v}$  and  $\frac{\partial\Phi}{\partial z} = \psi_1(x_1, x_2) + i\psi_2(x_1, x_2)$ . Thanks to the boundary condition (2.9), we have

$$\tilde{w}_1|_{\partial\Omega} = 2\partial_{\bar{z}}\tilde{v}|_{\partial\Omega} = (\nu_1 + i\nu_2)\frac{\partial\tilde{v}}{\partial\nu}|_{\partial\Omega}, \quad \tilde{w}_2|_{\partial\Omega} = 2\partial_z\tilde{v}|_{\partial\Omega} = (\nu_1 - i\nu_2)\frac{\partial\tilde{v}}{\partial\nu}|_{\partial\Omega}.$$

By Proposition 2.1

$$\begin{aligned} & \left\| \left( \frac{\partial}{\partial x_1} - i\psi_2\tau \right) \tilde{w}_1 \right\|_{L^2(\Omega)}^2 - \tau \int_{\partial\Omega} (\nabla\varphi_1, \nu) \left| \frac{\partial\tilde{v}}{\partial\nu} \right|^2 d\sigma + \operatorname{Re} \int_{\partial\Omega} i \left( \left( \nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2} \right) \tilde{w}_1 \right) \overline{\tilde{w}_1} d\sigma \\ & \quad + \left\| \left( i \frac{\partial}{\partial x_2} + \psi_1\tau \right) \tilde{w}_1 \right\|_{L^2(\Omega)}^2 = \|fe^{\tau\varphi_1}\|_{L^2(\Omega)}^2 \end{aligned}$$

and

$$\begin{aligned} & \left\| \left( \frac{\partial}{\partial x_1} + i\psi_2\tau \right) \tilde{w}_2 \right\|_{L^2(\Omega)}^2 - \tau \int_{\partial\Omega} (\nabla\varphi_1, \nu) \left| \frac{\partial\tilde{v}}{\partial\nu} \right|^2 d\sigma + \operatorname{Re} \int_{\partial\Omega} i \left( \left( -\nu_2 \frac{\partial}{\partial x_1} + \nu_1 \frac{\partial}{\partial x_2} \right) \tilde{w}_2 \right) \overline{\tilde{w}_2} d\sigma \\ & \quad + \left\| \left( i \frac{\partial}{\partial x_2} - \psi_1\tau \right) \tilde{w}_2 \right\|_{L^2(\Omega)}^2 = \|fe^{\tau\varphi_1}\|_{L^2(\Omega)}^2. \end{aligned}$$

Let us simplify the integral  $\operatorname{Re} i \int_{\partial\Omega} \left( \left( \nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2} \right) \tilde{w}_1 \right) \overline{\tilde{w}_1} d\sigma$ . We recall that  $\tilde{v} = ue^{\tau\varphi_1}$  and  $\tilde{w}_1 = (\nu_1 + i\nu_2)\frac{\partial\tilde{v}}{\partial\nu} = (\nu_1 + i\nu_2)\frac{\partial u}{\partial\nu}e^{\tau\varphi_1}$ . Thus

$$\begin{aligned} & \operatorname{Re} \int_{\partial\Omega} i \left( \left( \nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2} \right) \tilde{w}_1 \right) \overline{\tilde{w}_1} d\sigma = \\ & \operatorname{Re} \int_{\partial\Omega} i \left( \left( \nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2} \right) \left[ (\nu_1 + i\nu_2) \frac{\partial u}{\partial\nu} e^{\tau\varphi_1} \right] \right) (\nu_1 - i\nu_2) \frac{\partial u}{\partial\nu} e^{\tau\varphi_1} d\sigma = \\ & \operatorname{Re} \int_{\partial\Omega} i \left[ \left( \nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2} \right) (\nu_1 + i\nu_2) \right] \left| \frac{\partial\tilde{v}}{\partial\nu} \right|^2 (\nu_1 - i\nu_2) d\sigma + \\ & \operatorname{Re} \int_{\partial\Omega} \frac{1}{2} i \left( \nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2} \right) \left| \frac{\partial\tilde{v}}{\partial\nu} \right|^2 d\sigma = \int_{\partial\Omega} \left| \frac{\partial\tilde{v}}{\partial\nu} \right|^2 d\sigma. \end{aligned}$$

Let us simplify the integral  $\operatorname{Re} \int_{\partial\Omega} i \left( \left( -\nu_2 \frac{\partial}{\partial x_1} + \nu_1 \frac{\partial}{\partial x_2} \right) \tilde{w}_2 \right) \overline{\tilde{w}_2} d\sigma$ . We recall that  $\tilde{v} = ue^{\tau\varphi_1}$  and  $\tilde{w}_2 = (\nu_1 - i\nu_2) \frac{\partial \tilde{v}}{\partial \nu} = (\nu_1 - i\nu_2) \frac{\partial u}{\partial \nu} e^{\tau\varphi_1}$ . We conclude

$$\begin{aligned}
& \operatorname{Re} \int_{\partial\Omega} i \left( \left( -\nu_2 \frac{\partial}{\partial x_1} + \nu_1 \frac{\partial}{\partial x_2} \right) \tilde{w}_2 \right) \overline{\tilde{w}_2} d\sigma = \\
(2.11) \quad & \operatorname{Re} \int_{\partial\Omega} i \left( \left( -\nu_2 \frac{\partial}{\partial x_1} + \nu_1 \frac{\partial}{\partial x_2} \right) \left[ (\nu_1 - i\nu_2) \frac{\partial u}{\partial \nu} e^{\tau\varphi_1} \right] \right) (\nu_1 + i\nu_2) \frac{\partial u}{\partial \nu} e^{\tau\varphi_1} d\sigma = \\
& \operatorname{Re} \int_{\partial\Omega} i \left[ \left( -\nu_2 \frac{\partial}{\partial x_1} + \nu_1 \frac{\partial}{\partial x_2} \right) (\nu_1 - i\nu_2) \right] \left| \frac{\partial \tilde{v}}{\partial \nu} \right|^2 (\nu_1 + i\nu_2) d\sigma - \\
& \operatorname{Re} \int_{\partial\Omega} \frac{1}{2} i \left( \nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2} \right) \left| \frac{\partial \tilde{v}}{\partial \nu} \right|^2 d\sigma = \int_{\partial\Omega} \left| \frac{\partial \tilde{v}}{\partial \nu} \right|^2 d\sigma.
\end{aligned}$$

Using the above formulae we obtain

$$\begin{aligned}
(2.12) \quad & \left\| \left( \frac{\partial}{\partial x_1} + i\psi_2\tau \right) \tilde{w}_2 \right\|_{L^2(\Omega)}^2 + \left\| \left( i \frac{\partial}{\partial x_2} - \psi_1\tau \right) \tilde{w}_2 \right\|_{L^2(\Omega)}^2 - 2\tau \int_{\partial\Omega} (\nu, \nabla\varphi_1) \left| \frac{\partial \tilde{v}}{\partial \nu} \right|^2 d\sigma \\
& + \left\| \left( \frac{\partial}{\partial x_1} - i\psi_2\tau \right) \tilde{w}_1 \right\|_{L^2(\Omega)}^2 + \left\| \left( i \frac{\partial}{\partial x_2} + \psi_1\tau \right) \tilde{w}_1 \right\|_{L^2(\Omega)}^2 \\
& + 2 \int_{\partial\Omega} \left| \frac{\partial \tilde{v}}{\partial \nu} \right|^2 d\sigma = 2 \|f e^{\tau\varphi_1}\|_{L^2(\Omega)}^2.
\end{aligned}$$

Let a function  $\tilde{\psi}_k$  satisfy

$$\frac{\partial \tilde{\psi}_1}{\partial x_1} = \psi_2, \quad \frac{\partial \tilde{\psi}_2}{\partial x_2} = \psi_1 \quad \text{in } \Omega.$$

We can rewrite equality (2.12) in the form

$$\begin{aligned}
(2.13) \quad & \left\| \frac{\partial}{\partial x_1} (e^{i\tilde{\psi}_1\tau} \tilde{w}_2) \right\|_{L^2(\Omega)}^2 + \left\| \frac{\partial}{\partial x_2} (e^{i\tilde{\psi}_2\tau} \tilde{w}_2) \right\|_{L^2(\Omega)}^2 - 2\tau \int_{\partial\Omega} (\nu, \nabla\varphi_1) \left| \frac{\partial \tilde{v}}{\partial \nu} \right|^2 d\sigma \\
& + \left\| \frac{\partial}{\partial x_1} (e^{-i\tilde{\psi}_1\tau} \tilde{w}_1) \right\|_{L^2(\Omega)}^2 + \left\| \frac{\partial}{\partial x_2} (e^{-i\tilde{\psi}_2\tau} \tilde{w}_1) \right\|_{L^2(\Omega)}^2 \\
& + 2 \int_{\partial\Omega} \left| \frac{\partial \tilde{v}}{\partial \nu} \right|^2 d\sigma = 2 \|f e^{\tau\varphi_1}\|_{L^2(\Omega)}^2.
\end{aligned}$$

Observe that there exists some positive constant  $C > 0$ , independent of  $\tau$  such that

$$\begin{aligned}
(2.14) \quad & \frac{1}{C} (\|\tilde{w}_1\|_{L^2(\Omega)}^2 + \|\tilde{w}_2\|_{L^2(\Omega)}^2) \leq \frac{1}{2} \left\| \frac{\partial}{\partial x_1} (e^{i\tilde{\psi}_2\tau} \tilde{w}_2) \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \left\| \frac{\partial}{\partial x_2} (e^{i\tilde{\psi}_1\tau} \tilde{w}_2) \right\|_{L^2(\Omega)}^2 \\
& - \tau \int_{\partial\Omega_-} (\nu, \nabla\varphi_1) \left| \frac{\partial \tilde{v}}{\partial \nu} \right|^2 d\sigma \\
& + \frac{1}{2} \left\| \frac{\partial}{\partial x_1} (e^{-i\tilde{\psi}_1\tau} \tilde{w}_1) \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \left\| \frac{\partial}{\partial x_2} (e^{-i\tilde{\psi}_2\tau} \tilde{w}_1) \right\|_{L^2(\Omega)}^2.
\end{aligned}$$

Since  $\tilde{v}$  is a real-valued function, we have

$$\left\| \frac{\partial \tilde{v}}{\partial x_1} + \tau \psi_1 \tilde{v} \right\|_{L^2(\Omega)}^2 + \left\| \frac{\partial \tilde{v}}{\partial x_2} - \tau \psi_2 \tilde{v} \right\|_{L^2(\Omega)}^2 \leq C_0 (\|\tilde{w}_1\|_{L^2(\Omega)}^2 + \|\tilde{w}_2\|_{L^2(\Omega)}^2).$$

Therefore

$$(2.15) \quad \begin{aligned} & \left\| \frac{\partial \tilde{v}}{\partial x_1} \right\|_{L^2(\Omega)}^2 - \tau \int_{\Omega} \left( \frac{\partial \psi_1}{\partial x_1} - \frac{\partial \psi_2}{\partial x_2} \right) \tilde{v}^2 dx + \|\tau \psi_1 \tilde{v}\|_{L^2(\Omega)}^2 \\ & + \left\| \frac{\partial \tilde{v}}{\partial x_2} \right\|_{L^2(\Omega)}^2 + \|\tau \psi_2 \tilde{v}\|_{L^2(\Omega)}^2 \leq C_1 (\|\tilde{w}_1\|_{L^2(\Omega)}^2 + \|\tilde{w}_2\|_{L^2(\Omega)}^2). \end{aligned}$$

By the Cauchy-Riemann equations, the second term of the left hand side of (2.15) is zero.

Now since by assumption (2.2) the function  $\Phi$  has only non degenerate critical points, we have

$$(2.16) \quad \tau \|\tilde{v}\|_{L^2(\Omega)}^2 \leq C \left( \|\tilde{v}\|_{H^1(\Omega)}^2 + \tau^2 \left\| \left| \frac{\partial \Phi}{\partial z} \right| \tilde{v} \right\|_{L^2(\Omega)}^2 \right).$$

By (2.15) and (2.16)

$$(2.17) \quad \tau \|\tilde{v}\|_{L^2(\Omega)}^2 + \|\tilde{v}\|_{H^1(\Omega)}^2 + \tau^2 \left\| \left| \frac{\partial \Phi}{\partial z} \right| \tilde{v} \right\|_{L^2(\Omega)}^2 \leq C_1 (\|\tilde{w}_1\|_{L^2(\Omega)}^2 + \|\tilde{w}_2\|_{L^2(\Omega)}^2).$$

Using (2.17), we obtain from (2.13) and (2.14) that

$$\begin{aligned} & \frac{1}{C_5} \left( \tau \|\tilde{v}\|_{L^2(\Omega)}^2 + \|\tilde{v}\|_{H^1(\Omega)}^2 + \tau^2 \left\| \left| \frac{\partial \Phi}{\partial z} \right| \tilde{v} \right\|_{L^2(\Omega)}^2 \right) - 2\tau \int_{\partial\Omega} (\nu, \nabla \varphi_1) \left| \frac{\partial \tilde{v}}{\partial \nu} \right|^2 d\sigma \\ & + 2 \int_{\partial\Omega} \left| \frac{\partial \tilde{v}}{\partial \nu} \right|^2 d\sigma \leq 2 \|f e^{s\varphi_1}\|_{L^2(\Omega)}^2 - \tau \int_{\partial\Omega_-} (\nu, \nabla \varphi_1) \left| \frac{\partial \tilde{v}}{\partial \nu} \right|^2 d\sigma \end{aligned}$$

concluding the proof of the theorem.  $\square$

We note that in the theorem we can add a zeroth order term to the Laplacian and the estimate is valid for large enough  $\tau$ .

As usual the Carleman estimate implies the existence of solutions for the Schrödinger equation satisfying estimates with appropriate weights.

Consider the following problem

$$(2.18) \quad \Delta u + q_0 u = f \quad \text{in } \Omega, \quad u|_{\tilde{\Gamma}} = g,$$

where  $\tilde{\Gamma} \subset \{x \in \partial\Omega | (\nu, \nabla \varphi_1) < 0\}$ . We have

**Proposition 2.2.** *Let  $q_0 \in L^\infty(\Omega)$ . There exists  $\tau_0 > 0$  such that for all  $\tau > \tau_0$  there exists a solution to problem (2.18) such that*

$$(2.19) \quad \|ue^{-\tau\varphi_1}\|_{L^2(\Omega)} \leq C (\|fe^{-\tau\varphi_1}\|_{L^2(\Omega)} + \|ge^{-\tau\varphi_1}\|_{L^2(\tilde{\Gamma})}) / \tau^{\frac{1}{2}},$$

*Proof.* Let us introduce the space

$$H = \left\{ v \in H_0^1(\Omega) \mid \Delta v + q_0 v \in L^2(\Omega), \frac{\partial v}{\partial \nu} \Big|_{\partial\Omega_+} = 0 \right\}$$

with the scalar product

$$(v_1, v_2)_H = \int_{\Omega} e^{2\tau\varphi_1} (\Delta v_1 + q_0 v_1) (\Delta v_2 + q_0 v_2) dx.$$

By Proposition 2.1  $H$  is a Hilbert space. Consider the linear functional on  $H : v \rightarrow \int_{\Omega} v f dx + \int_{\bar{\Gamma}} g \frac{\partial v}{\partial \nu} d\sigma$ . By (2.10) this is the continuous linear functional with the norm estimated by a constant  $C(\|f e^{\tau\varphi_1}\|_{L^2(\Omega)} + \|g e^{\tau\varphi_1}\|_{L^2(\bar{\Gamma})})/\tau^{\frac{1}{2}}$ . Therefore by the Riesz theorem there exists an element  $\hat{v} \in H$  so that

$$\int_{\Omega} v f dx + \int_{\bar{\Gamma}} g \frac{\partial v}{\partial \nu} d\sigma = \int_{\Omega} e^{2\tau\varphi_1} (\Delta \hat{v} + q_0 \hat{v}) (\Delta v + q_0 v) dx.$$

Then, as a solution to (2.18), we take the function  $u = e^{2\tau\varphi_1} (\Delta \hat{v} + q_0 \hat{v})$ .  $\square$

### 3. Complex geometrical optics solutions with degenerate weights

In this section we construct the complex geometrical optics solutions which will play the critical role in the proof of Theorem 1.1.

We first observe that we can put the sets  $\Gamma_-$  and  $\partial\Omega \setminus \Gamma_{-, \epsilon}$  in a more convenient position on the boundary of the unit ball and slightly deform the ball itself.

Namely we set

$$(3.1) \quad \Omega \subset B(0, 1), \quad \Gamma_- \subset S^1, \quad \mathcal{S} \equiv \partial\Omega \setminus \Gamma_{-, \epsilon} \subset S^1.$$

Let  $\ell_+ \in \Gamma_+$  be a piece of  $\partial\Omega$  between the points  $\hat{x}_+$  and  $\hat{x}_{+, \epsilon}$  and  $\ell_- \in \Gamma_+$  be a piece of  $\partial\Omega$  between the points  $\hat{x}_-$  and  $\hat{x}_{-, \epsilon}$ . Then

$$(3.2) \quad \ell_{\pm} \subset B(0, 1).$$

We construct complex geometrical optics solutions of the Schrödinger equation  $\Delta + q_1$ , with  $q_1$  satisfying the conditions of Theorem 1.1. Consider the equation

$$(3.3) \quad L_1 u = \Delta u + q_1 u = 0 \quad \text{in } \Omega.$$

Let  $\Phi(z)$  be a holomorphic function satisfying (2.1) and (2.2). Let us fix small positive constants  $\epsilon, \epsilon'$  and consider two domains:

$$(3.4) \quad \partial\Omega_{-, \epsilon} = \{x \in \partial\Omega \mid (\nabla\varphi_1, \nu) < -\epsilon\}, \quad \partial\Omega_{+, \epsilon'} = \{x \in \partial\Omega \mid (\nabla\varphi_1, \nu) > \epsilon'\}.$$

Suppose that

$$(3.5) \quad \bar{\Gamma}_- \subset \partial\Omega_{-, \epsilon},$$

and

$$(3.6) \quad \bar{\mathcal{S}} \subset \partial\Omega_{+, \epsilon'}.$$

We will construct solutions to (3.3) of the form

$$(3.7) \quad u_1(x) = e^{\tau\Phi(z)}a(z) - \chi_1(x)e^{\tau\Phi(\frac{1}{\bar{z}})}a\left(\frac{1}{\bar{z}}\right) + e^{\tau\Phi}u_{11} + e^{\tau\varphi_1}u_{12}, \quad u_1|_{\Gamma_-} = 0.$$

We explain in the next subsections the different phase function  $\varphi_1$  and the amplitude  $a(z)$  in (3.7). Moreover we derive the behavior for large  $\tau$  of the different pieces of the complex geometrical optics solutions.

**3.1. The amplitude  $a(z)$  and the function  $\chi_1$ .** The amplitude  $a(z)$  has the following properties:

$$a \in C^2(\bar{\Omega}), \quad \frac{\partial a}{\partial \bar{z}} \equiv 0, \quad a(z) \neq 0 \text{ on } \bar{\Omega}.$$

Next we construct the cut-off function  $\chi_1(x)$ .

By (3.1) and (3.2), there exists a neighborhood  $\mathcal{O}_1$  of the set  $\Gamma_-$  such that  $\tilde{\varphi}_1(x) = \operatorname{Re} \Phi(\frac{1}{\bar{z}})$  is a harmonic function satisfying

$$(3.8) \quad \tilde{\varphi}_1(x) < \varphi(x), \quad \forall x \in \Omega \cap \mathcal{O}_1,$$

$$(3.9) \quad \partial\Omega \cap \mathcal{O}_1 \subset \partial\Omega_{-, -\frac{\epsilon}{2}},$$

$$(3.10) \quad \operatorname{supp} \nabla \chi_1 \subset\subset B(0, 1) \cap \mathcal{O}_1.$$

Consider the following integral

$$J(\tau) = \int_{\Omega} \chi_1 r(x) e^{\tau\Phi(\frac{1}{\bar{z}}) - \tau\overline{\Phi(z)}} dx.$$

We have

**Proposition 3.1.** *Let  $r \in C^{1+\ell}(\bar{\Omega})$  for some positive  $\ell$ . Then*

$$J(\tau) = o\left(\frac{1}{\tau}\right).$$

*Proof.* Observe that the function  $\chi_1$  can be chosen in such a way that

$$(3.11) \quad \partial_{\bar{z}} \left( \Phi\left(\frac{1}{\bar{z}}\right) - \overline{\Phi(z)} \right) |_{\operatorname{supp} \chi_1} \neq 0.$$

Assume that for some point from  $\partial\Omega_{-, -\epsilon}$  we have

$$\partial_{\bar{z}} \left( \Phi\left(\frac{1}{\bar{z}}\right) - \overline{\Phi(z)} \right) |_{\operatorname{supp} \chi_1} = 0,$$

and the above equality is equivalent to

$$\operatorname{Re}(\Phi'(z)z) = 0.$$

This equality and the Cauchy-Riemann equations imply that  $\frac{\partial \varphi}{\partial \nu} = 0$  at this point, which is a contradiction. Since it suffices to choose  $\operatorname{supp} \chi_1$  close to  $\Gamma_-$ , the proof of (3.11) is completed.

Therefore

$$J(\tau) = \int_{\Omega} \chi_1 r(x) e^{\tau\Phi(\frac{1}{\bar{z}}) - \tau\overline{\Phi(z)}} dx = \frac{1}{\tau} \int_{\Omega} \chi_1 r(x) \frac{1}{\partial_{\bar{z}}(\Phi(\frac{1}{\bar{z}}) - \overline{\Phi(z)})} \partial_{\bar{z}} e^{\tau\Phi(\frac{1}{\bar{z}}) - \tau\overline{\Phi(z)}} dx.$$

Integrating by parts we have:

$$\begin{aligned} J(\tau) &= -\frac{1}{\tau} \int_{\Omega} \partial_{\bar{z}}(\chi_1 r(x)) \frac{1}{\partial_{\bar{z}}(\Phi(\frac{1}{z}) - \overline{\Phi(z)})} e^{\tau\Phi(\frac{1}{z}) - \tau\overline{\Phi(z)}} dx \\ &+ \frac{1}{2\tau} \int_{\partial\Omega} \chi_1 r(x) \frac{1}{\partial_{\bar{z}}(\Phi(\frac{1}{z}) - \overline{\Phi(z)})} (\nu_1 + i\nu_2) e^{\tau\Phi(\frac{1}{z}) - \tau\overline{\Phi(z)}} d\sigma = J_1 + J_2. \end{aligned}$$

Observe that on  $\partial\Omega$

$$e^{\tau\Phi(\frac{1}{z}) - \tau\overline{\Phi(z)}} = e^{2\tau i \operatorname{Im} \Phi(z)}.$$

Using a stationary phase and taking into account that  $\partial_{\nu} \operatorname{Re} \Phi = \partial_{\tau} \operatorname{Im} \Phi \neq 0$  on  $\operatorname{supp} \chi_1 \cap \partial\Omega$ , we obtain

$$J_2 = o\left(\frac{1}{\tau}\right).$$

Next we observe that since  $r \in C^{1+\ell}(\overline{\Omega})$  we have

$$J_1 = o\left(\frac{1}{\tau}\right).$$

The proof of the proposition is finished.  $\square$

**3.2. Construction of  $u_{11}$ .** The function  $e^{\tau\Phi(z)}a(z) - \chi_1(x)e^{\tau\Phi(\frac{1}{z})}a(\frac{1}{z})$  does not satisfy equation (3.3). We construct  $u_{11}$  in the next term in the asymptotic expansion. Before we start the construction of this term we need several propositions.

Let us introduce the operators:

$$\begin{aligned} \partial_{\bar{z}}^{-1}g &= \frac{1}{2\pi i} \int_{\Omega} \frac{g(\zeta, \bar{\zeta})}{\zeta - z} d\zeta \wedge d\bar{\zeta} = -\frac{1}{\pi} \int_{\Omega} \frac{g(\zeta, \bar{\zeta})}{\zeta - z} d\xi_1 d\xi_2, \\ \partial_z^{-1}g &= -\frac{1}{2\pi i} \int_{\Omega} \frac{\bar{g}(\zeta, \bar{\zeta})}{\zeta - z} d\zeta \wedge d\bar{\zeta} = -\frac{1}{\pi} \int_{\Omega} \frac{g(\zeta, \bar{\zeta})}{\bar{\zeta} - \bar{z}} d\xi_1 d\xi_2. \end{aligned} \tag{3.12}$$

Then we know (e.g., [26] p. 56):

**Proposition 3.2.** *Let  $m \geq 0$  be an integer number and  $\alpha \in (0, 1)$ . The operators  $\partial_{\bar{z}}^{-1}, \partial_z^{-1} \in \mathcal{L}(C^{m+\alpha}(\overline{\Omega}), C^{m+\alpha+1}(\overline{\Omega}))$ .*

Here and henceforth  $\mathcal{L}(X, Y)$  denotes the Banach space of all bounded linear operators from a Banach space  $X$  to another Banach space  $Y$ .

We define two other operators:

$$R_{\Phi}g = e^{\tau(\overline{\Phi(z)} - \Phi(z))} \partial_{\bar{z}}^{-1}(g e^{\tau(\Phi(z) - \overline{\Phi(z)})}), \quad \tilde{R}_{\Phi}g = e^{\tau(\overline{\Phi(z)} - \Phi(z))} \partial_z^{-1}(g e^{\tau(\Phi(z) - \overline{\Phi(z)})}). \tag{3.13}$$

**Proposition 3.3.** *Let  $g \in C^{\epsilon}(\overline{\Omega})$  for some positive  $\epsilon$ . The function  $R_{\Phi}g$  is a solution to*

$$\partial_{\bar{z}} R_{\Phi}g - \tau \frac{\partial \overline{\Phi(z)}}{\partial \bar{z}} R_{\Phi}g = g \quad \text{in } \Omega. \tag{3.14}$$

The function  $\tilde{R}_{\Phi}g$  solves

$$\partial_z \tilde{R}_{\Phi}g + \tau \frac{\partial \Phi(z)}{\partial z} \tilde{R}_{\Phi}g = g \quad \text{in } \Omega. \tag{3.15}$$

*Proof.* The proof is by direct computations:

$$\begin{aligned} \partial_z \tilde{R}_\Phi g + \tau \frac{\partial \Phi(z)}{\partial z} \tilde{R}_\Phi g &= \partial_z (e^{\tau(\overline{\Phi(z)} - \Phi(z))} \partial_z^{-1} (g e^{\tau(\Phi(z) - \overline{\Phi(z)})})) \\ &\quad + \tau \frac{\partial \Phi(z)}{\partial z} (e^{\tau(\overline{\Phi(z)} - \Phi(z))} \partial_z^{-1} (g e^{\tau(\Phi(z) - \overline{\Phi(z)})})) = \\ -\tau \frac{\partial \Phi(z)}{\partial z} (e^{\tau(\overline{\Phi(z)} - \Phi(z))} \partial_z^{-1} (g e^{\tau(\Phi(z) - \overline{\Phi(z)})})) &+ (e^{\tau(\overline{\Phi(z)} - \Phi(z))} (g e^{\tau(\Phi(z) - \overline{\Phi(z)})})) \\ &\quad + \tau \frac{\partial \Phi(z)}{\partial z} (e^{\tau(\overline{\Phi(z)} - \Phi(z))} \partial_z^{-1} (g e^{\tau(\Phi(z) - \overline{\Phi(z)})})) = g. \end{aligned}$$

□

Denote

$$\mathcal{O}_\epsilon = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) \leq \epsilon\}.$$

**Proposition 3.4.** *Let  $g \in C^1(\Omega)$ ,  $g|_{\mathcal{O}_\epsilon} \equiv 0$ ,  $g(x) \neq 0$  for all  $x \in \mathcal{H}$ . Then*

$$(3.16) \quad |R_\Phi g(x)| + |\tilde{R}_\Phi g(x)| \leq C \max_{x \in \mathcal{H}} |g(x)| / \tau$$

for all  $x \in \mathcal{O}_{\epsilon/2}$ . If  $g \in C^2(\overline{\Omega})$  and  $g|_{\mathcal{H}} = 0$  then

$$(3.17) \quad |R_\Phi g(x)| + |\tilde{R}_\Phi g(x)| \leq C / \tau^2$$

for all  $x \in \mathcal{O}_{\epsilon/2}$ .

*Proof.* Observe that  $e^{\tau(\Phi(z) - \overline{\Phi(z)})} = e^{2i\tau \text{Im}\Phi(z)}$ . By the Cauchy-Riemann equations, the sets of the critical points of  $\Phi(z)$  and  $\text{Im}\Phi(z)$  are exactly the same. Therefore by our assumptions the Hessian of  $\text{Im}\Phi(z)$  is nondegenerate at each point of  $\mathcal{H}$  and it is enough to show that

$$\left| \int_{\Omega} e^{2i\tau \text{Im}\Phi(z)} \frac{g(\zeta, \bar{\zeta})}{z - \zeta} d\zeta \wedge d\bar{\zeta} \right| \leq C \max_{x \in \mathcal{H}} |g(x)| / \tau \quad \text{and} \quad \left| \int_{\Omega} e^{2i\tau \text{Im}\Phi(z)} \frac{g(\zeta, \bar{\zeta})}{z - \zeta} d\zeta \wedge d\bar{\zeta} \right| \leq C / \tau^2.$$

We observe that for any  $z = x_1 + ix_2 \in \mathcal{O}_{\frac{\epsilon}{2}}$  the function  $\frac{g(\zeta)}{z - \zeta}$  in the variable  $\zeta$  is smooth and compactly supported. The statement of the proposition follows from the standard stationary phase argument (see e.g., [14]). □

Denote

$$(3.18) \quad r(z) = \prod_{k=1}^{\ell} (z - z_k) \quad \text{where } \mathcal{H} = \{z_1, \dots, z_\ell\}.$$

**Proposition 3.5.** *Let  $g \in C^1(\overline{\Omega})$ ,  $g|_{\mathcal{O}_\epsilon} \equiv 0$ . Then for each  $\delta \in (0, 1)$  there exists a constant  $C(\delta)$  such that*

$$(3.19) \quad \|\tilde{R}_\Phi(\overline{r(z)}g)\|_{L^2(\Omega)} \leq C(\delta) \|g\|_{C^1(\overline{\Omega})} / \tau^{1-\delta}, \quad \|R_\Phi(r(z)g)\|_{L^2(\Omega)} \leq C(\delta) \|g\|_{C^1(\overline{\Omega})} / \tau^{1-\delta}.$$

*Proof.* Denote  $v = \tilde{R}_\Phi(\overline{r(z)}g)$ . By Proposition 3.4

$$(3.20) \quad \|v\|_{L^2(\mathcal{O}_{\epsilon/2})} \leq C / \tau.$$

Then by Proposition 3.3

$$\frac{\partial v}{\partial z} + \tau \frac{\partial \Phi}{\partial z} v = \overline{r(z)}g \quad \text{in } \Omega.$$

There exists a function  $p$  such that

$$-\frac{\partial p}{\partial \bar{z}} + \tau \frac{\overline{\partial \Phi(z)}}{\partial z} p = v \quad \text{in } \Omega$$

and there exists a constant  $C > 0$  independent of  $\tau$  such that

$$(3.21) \quad \|p\|_{L^2(\Omega)} \leq C \|v\|_{L^2(\Omega)}.$$

Let  $\chi$  be a nonnegative function such that  $\chi \equiv 0$  on  $\mathcal{O}_{\frac{\epsilon}{16}}$  and  $\chi \equiv 1$  on  $\Omega \setminus \mathcal{O}_{\frac{\epsilon}{8}}$ . Setting  $\tilde{p} = \chi p$  and using  $g|_{\mathcal{O}_\epsilon} \equiv 0$ , we have that

$$\int_{\Omega} \overline{r(z)} g \bar{p} dx = \int_{\Omega \setminus \mathcal{O}_\epsilon} \overline{r(z)} g \bar{p} dx = \int_{\Omega} \overline{r(z)} g \tilde{p} dx$$

and

$$(3.22) \quad -\frac{\partial \tilde{p}}{\partial \bar{z}} + \tau \frac{\overline{\partial \Phi(z)}}{\partial z} \tilde{p} = \chi v - p \frac{\partial \chi}{\partial \bar{z}} \quad \text{in } \Omega.$$

Then

$$(3.23) \quad \|\chi^{\frac{1}{2}} v\|_{L^2(\Omega)}^2 = \int_{\Omega} \overline{r(z)} g \tilde{p} dx + \int_{\Omega} p \frac{\partial \chi}{\partial \bar{z}} \bar{v} dx.$$

Note that

$$(3.24) \quad \|\tilde{p}\|_{H^1(\Omega)} \leq C\tau \|p\|_{L^2(\Omega)} \leq C\tau \|v\|_{L^2(\Omega)}, \quad \int_{\Omega} \overline{r(z)} g \tilde{p} dx = \int_{\Omega} \overline{gr(z)} \tilde{p} dx.$$

Taking the scalar product of (3.22) and  $\frac{\overline{r(z)}}{\partial_z \Phi(z)} g$  we obtain

$$\begin{aligned} \int_{\Omega} \frac{\overline{r(z)}}{\partial_z \Phi(z)} g \left( -\frac{\partial \tilde{p}}{\partial \bar{z}} + \tau \frac{\overline{\partial \Phi(z)}}{\partial z} \tilde{p} \right) dx &= \int_{\Omega} \frac{\overline{r(z)}}{\partial_z \Phi(z)} g \left( \chi v - p \frac{\partial \chi}{\partial \bar{z}} \right) dx, \\ \tau \int_{\Omega} \overline{gr(z)} \tilde{p} dx &= \int_{\Omega} \frac{\overline{r(z)}}{\partial_z \Phi(z)} g \left( \chi v + p \frac{\partial \chi}{\partial \bar{z}} \right) dx - \int_{\Omega} \frac{\partial}{\partial z} \left( \frac{\overline{r(z)}}{\partial_z \Phi(z)} g \right) \tilde{p} dx. \end{aligned}$$

By (3.24) and the Sobolev embedding theorem, for each  $\epsilon \in (0, \frac{1}{2})$  we have

$$(3.25) \quad \left| \int_{\Omega} \frac{\partial}{\partial z} \left( \frac{\overline{r(z)}}{\partial_z \Phi(z)} g \right) \tilde{p} dx \right| \leq \left| \int_{\Omega} \frac{\overline{r(z)} \partial_z^2 \Phi(z)}{(\partial_z \Phi(z))^2} g \tilde{p} dx \right| + \left| \int_{\Omega} \frac{\overline{r(z)}}{\partial_z \Phi(z)} \frac{\partial g}{\partial z} \tilde{p} dx \right| \\ \leq C \|g\|_{C^1(\bar{\Omega})} \left\| \frac{1}{\partial_z \Phi(z)} \right\|_{L^{2-\epsilon}(\Omega)} \|\tilde{p}\|_{L^{\frac{2}{1-\epsilon}}(\Omega)} \leq C \|\tilde{p}\|_{H^{\delta_3(\epsilon)}(\Omega)} \leq C\tau^{\delta_4} \|v\|_{L^2(\Omega)}.$$

Here we choose  $\delta_3(\epsilon) > 0$  such that  $\delta_3(\epsilon) \rightarrow +0$  as  $\epsilon \rightarrow +0$  and  $H^{\delta_3(\epsilon)}(\Omega) \subset L^{\frac{2}{1-\epsilon}}(\Omega)$ . Therefore

$$(3.26) \quad \left| \int_{\Omega} \overline{gr(z)} \tilde{p} dx \right| \leq C\tau^{-1+\delta_4} \|v\|_{L^2(\Omega)} \quad \text{as } \delta_4 \rightarrow +0.$$

By (3.20)

$$(3.27) \quad \left| \int_{\Omega} p \frac{\partial \chi}{\partial \bar{z}} \bar{v} dx \right| \leq C \|p\|_{L^2(\Omega)} \|v\|_{L^2(\mathcal{O}_{\frac{\epsilon}{8}})} \leq C \|p\|_{L^2(\Omega)} / \tau.$$

By (3.21), (3.26) and (3.27) we obtain from (3.23)

$$\|v\|_{L^2(\Omega)}^2 \leq C(\tau^{-1+\delta_4}\|v\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)}/\tau) \leq C\tau^{-1+\delta_4}\|v\|_{L^2(\Omega)}.$$

In the last estimate we used (3.21).  $\square$

We construct the function  $u_{11}$  in the form

$$(3.28) \quad u_{11} = (u_{11,1} + u_{11,2}),$$

where the functions  $u_{11,k}$  are defined in the following way: Let  $e_i \in C^\infty(\overline{\Omega})$ ,  $e_1 + e_2 \equiv 1$ ,  $e_2$  is zero in some neighborhood of  $\mathcal{H}$  and  $e_1$  is zero in a neighborhood of  $\partial\Omega$ . The second term  $u_{11}$  in the asymptotic (3.7), is constructed to satisfy

$$(3.29) \quad \Delta u_{11} + 4\tau \frac{\partial\Phi(z)}{\partial z} \partial_{\bar{z}} u_{11} = aq_1 + o\left(\frac{1}{\tau}\right) \quad \text{in } \Omega.$$

Let  $m_1(z), m_2(z), m_3(z)$  be polynomials satisfying

$$\begin{aligned} (\partial_{\bar{z}}^{-1}(aq_1) - m_1(z))|_{\mathcal{H}} &= 0, \\ m_2(z)|_{\mathcal{H}} &= 0, \quad (\partial_z(\partial_{\bar{z}}^{-1}(aq_1) - m_1(z)) - m_2(z))|_{\mathcal{H}} = 0, \\ m_3(z)|_{\mathcal{H}} &= \partial_z m_3(z)|_{\mathcal{H}} = 0, \quad \partial_z^2(\partial_{\bar{z}}^{-1}(aq_1) - m_1(z) - m_2(z) - m_3(z))|_{\mathcal{H}} = 0. \end{aligned}$$

The equation for  $u_{11}$  can be transformed into

$$4\partial_z u_{11} + 4\tau \frac{\partial\Phi(z)}{\partial z} u_{11} = \partial_{\bar{z}}^{-1}(aq_1) - \sum_{k=1}^3 m_k(z) + o\left(\frac{1}{\tau}\right) \quad \text{in } \Omega.$$

Then

$$4\partial_z u_{11,1} + 4\tau \frac{\partial\Phi(z)}{\partial z} u_{11,1} = e_1 \left( \partial_{\bar{z}}^{-1}(aq_1) - \sum_{k=1}^3 m_k(z) \right) \quad \text{in } \Omega.$$

and we define  $u_{11,1}$  as

$$(3.30) \quad u_{11,1}(x) = \frac{1}{4} \tilde{R}_\Phi \left( e_1 \left( \partial_{\bar{z}}^{-1}(aq_1) - \sum_{k=1}^3 m_k(z) \right) \right)$$

and we define  $u_{11,2}$  as

$$(3.31) \quad u_{11,2}(x) = \frac{1}{4} e_2(x) \left( \partial_{\bar{z}}^{-1}(aq_1) - \sum_{k=1}^3 m_k(z) \right) / (\tau \partial_z \Phi(z)).$$

Since by the assumption the function  $e_2$  vanishes near the zeros of  $\Phi$ , the function  $u_{11,2}$  is smooth.

We will apply Proposition 3.5 to the function  $u_{11,1}$  to obtain the asymptotic behavior in  $\tau$ . In order to do that we need to represent the function

$$(3.32) \quad \mathcal{G}_1 = e_1 \left( \partial_{\bar{z}}^{-1}(aq_1) - \sum_{k=1}^3 m_k(z) \right)$$

in the form

$$\mathcal{G}_1 = \overline{r(z)} g(x),$$

where  $g$  is some function from  $C^1(\overline{\Omega})$ . This is an equivalent representation of the function  $m = \partial_{\bar{z}}^{-1}(aq_1) - \sum_{k=1}^3 m_k(z)$  in the form

$$m = \overline{r(z)}g_1, \quad g_1 \in C^1(\overline{\Omega}).$$

We remind that the polynomial  $r(z)$  is given by (3.18). Denote as  $p = \partial_{\bar{z}}^{-1}(aq_1)$ . Let  $x_j$  be a critical point of the function  $Im\Phi$  and  $z_j \in \mathcal{H}$  (see (3.18)). By Taylor's formula  $p(x) = p(z_j) + p_1(z - z_j) + p_2(\bar{z} - \bar{z}_j) + p_{11}(z - z_j)^2 + p_{12}(z - z_j)(\bar{z} - \bar{z}_j) + p_{22}(\bar{z} - \bar{z}_j)^2 + q(z, \bar{z})$ . Then  $m = p_2(\bar{z} - \bar{z}_j) + p_{22}(\bar{z} - \bar{z}_j)^2 + p_{12}(z - z_j)(\bar{z} - \bar{z}_j) + q(z, \bar{z})$  and we set  $g_1 = (p_2(\bar{z} - \bar{z}_j) + p_{22}(\bar{z} - \bar{z}_j)^2 + p_{12}(z - z_j)(\bar{z} - \bar{z}_j) + q(z, \bar{z}))/\overline{r(z)}$ . Let us show that  $g_1 \in C^1(\overline{\Omega})$ . Obviously  $(p_2(\bar{z} - \bar{z}_j) + p_{22}(\bar{z} - \bar{z}_j)^2 + p_{12}(z - z_j)(\bar{z} - \bar{z}_j))/\overline{r(z)}$  is a smooth function and  $\tilde{q}(z, \bar{z}) = q(z, \bar{z})/\overline{r(z)}$  is of  $C^1$  outside of  $z = 0$ . Continue the function  $\tilde{q}$  by zero on  $z = 0$ . Since  $q = o(|z|^3)$  the partial derivatives of this function vanishes at zero.

By Proposition 3.5

$$(3.33) \quad \|u_{11,1}\|_{L^2(\Omega)} \leq C(\delta)/\tau^{1-\delta} \quad \forall \delta \in (0, 1).$$

**3.3. Construction of  $u_{12}$ .** We will define  $u_{12}$  as a solution to the inhomogeneous problem

$$(3.34) \quad \Delta(u_{12}e^{\tau\varphi_1}) + q_1u_{12}e^{\tau\varphi_1} = (q_1u_{11} + \Delta u_{11,2})e^{\tau\Phi} - L_1 \left( \chi_1 e^{\tau\Phi(\frac{1}{z})} a \left( \frac{1}{z} \right) \right) \quad \text{in } \Omega,$$

$$(3.35) \quad u_{12}|_{\Gamma_-} = u_{11}e^{\tau Im \Phi}.$$

This can be done since

$$\|q_1u_{11} + \Delta u_{11,2}\|_{L^2(\Omega)} \leq C(\delta)/\tau^{1-\delta} \quad \forall \delta \in (0, 1)$$

and by (3.8), (3.10)

$$\left\| L_1 \left( \chi_1 e^{\tau\Phi(\frac{1}{z})} a \left( \frac{1}{z} \right) \right) e^{-\tau\varphi_1} \right\|_{L^2(\Omega)} = o \left( \frac{1}{\tau^2} \right).$$

and by (3.16), (3.30), (3.31)

$$\|u_{11}\|_{C^0(\partial\Omega)} \leq \frac{C}{\tau}.$$

By Proposition 2.2 there exists a solution to (3.34) satisfying

$$(3.36) \quad \|u_{12}\|_{L^2(\Omega)} \leq C(\delta)/\tau^{\frac{3}{4}-\delta}, \quad \forall \delta \in (0, 1).$$

**3.4. Replacing  $\Phi$  by  $-\overline{\Phi}$ .** Now we construct complex geometrical optics solutions for the potential  $q_2$  satisfying the conditions of the Theorem 1.1 but with  $\Phi$  replaced by  $-\overline{\Phi}$  and the solution vanishes on  $\mathcal{S}$ .

This is very similar to what we have already done.

Consider the Schrödinger equation

$$(3.37) \quad L_2v = \Delta v + q_2v = 0 \quad \text{in } \Omega.$$

We will construct solutions to (3.37) of the form

$$(3.38) \quad v_1(x) = e^{-\tau\overline{\Phi(z)}b(z)} - \chi_1(x)e^{-\tau\overline{\Phi(\frac{1}{z})}b\left(\frac{1}{z}\right)} + e^{-\tau\overline{\Phi}}v_{11} + e^{-\tau\varphi_1}v_{12}, \quad v_1|_{\mathcal{S}} = 0.$$

The construction of  $v_1$  repeats the corresponding steps of the construction of  $u_1$ . In fact the only difference is that the parameter  $\tau$  is negative or in terms of the weight function we use  $-\varphi_1$  instead of  $\varphi_1$ . We provide the details for the sake of completeness. The amplitude  $b(z)$  has the following properties:

$$b \in C^2(\bar{\Omega}), \quad \frac{\partial b}{\partial \bar{z}} \equiv 0, \quad b(z) \neq 0 \text{ on } \bar{\Omega}.$$

Next we construct the cut-off function  $\chi_2(x)$  with  $\text{supp } \chi_2 \in \mathcal{O}_2$  where  $\mathcal{O}_2$  is a neighborhood of  $\mathcal{S} = \partial\Omega \setminus \Gamma_{-, \epsilon}$ , and

$$(3.39) \quad \tilde{\varphi}_1(x) > \varphi(x), \quad \forall x \in \Omega \cap \mathcal{O}_2,$$

$$(3.40) \quad \partial\Omega \cap \mathcal{O}_2 \subset \partial\Omega_{+, \frac{\epsilon'}{2}},$$

$$(3.41) \quad \text{supp } \nabla \chi_2 \subset\subset B(0, 1) \cap \mathcal{O}_2,$$

$$(3.42) \quad \text{supp } \chi_2 \cap \text{supp } \chi_1 = \emptyset.$$

Consider the following integral

$$\tilde{J}(\tau) = \int_{\Omega} \chi_2 r(x) e^{-\tau \overline{\Phi(\frac{1}{z})} + \tau \Phi(z)} dx.$$

Similarly to Proposition 3.1 we have

**Proposition 3.6.** *Let  $r \in C^{1+\ell}(\bar{\Omega})$  for some positive  $\ell$ . Then*

$$\tilde{J}(\tau) = o\left(\frac{1}{\tau}\right).$$

Now we construct  $v_{11}$ . Let  $e_i \in C^\infty(\bar{\Omega})$ ,  $e_1(x) + e_2(x) \equiv 1$ ,  $e_2$  is zero on some neighborhood of  $\mathcal{H}$  and  $e_1$  is zero on some neighborhood of  $\partial\Omega$ . Then

$$\Delta v_{11} - 4\tau \frac{\overline{\partial\Phi(z)}}{\partial z} \partial_z v_{11} = \bar{b}q_2 + o\left(\frac{1}{\tau}\right).$$

Let  $\tilde{m}_1(\bar{z}), \tilde{m}_2(\bar{z}), \tilde{m}_3(\bar{z})$  be polynomials satisfying

$$(\partial_z^{-1}(\bar{b}q_2) - \tilde{m}_1(\bar{z}))|_{\mathcal{H}} = 0,$$

$$\tilde{m}_2(\bar{z})|_{\mathcal{H}} = 0, \quad (\partial_{\bar{z}}(\partial_z^{-1}(\bar{b}q_2) - \tilde{m}_1(\bar{z})) - \tilde{m}_2(\bar{z}))|_{\mathcal{H}} = 0$$

and

$$\tilde{m}_3(\bar{z})|_{\mathcal{H}} = \partial_{\bar{z}} \tilde{m}_3(\bar{z})|_{\mathcal{H}} = 0, \quad \partial_{\bar{z}}^2(\partial_z^{-1}(\bar{b}q_2) - \tilde{m}_1(\bar{z}) - \tilde{m}_2(\bar{z}) - \tilde{m}_3(\bar{z}))|_{\mathcal{H}} = 0.$$

The equation for  $v_{11}$  can be transformed into

$$4\partial_{\bar{z}} v_{11} - 4\tau \frac{\overline{\partial\Phi(z)}}{\partial z} v_{11} = \left( \partial_z^{-1}(\bar{b}q_2) - \sum_{k=1}^3 \tilde{m}_k(\bar{z}) \right) + o\left(\frac{1}{\tau}\right).$$

Let

$$(3.43) \quad v_{11} = v_{11,1} + v_{11,2}.$$

Then

$$4\partial_{\bar{z}}v_{11,1} - 4\tau\frac{\overline{\partial\Phi(z)}}{\partial z}v_{11,1} = e_1 \left( \partial_z^{-1}(\bar{b}q_2) - \sum_{k=1}^3 \tilde{m}_k(\bar{z}) \right) \quad \text{in } \Omega,$$

and we take  $v_{11,1}$  as

$$(3.44) \quad v_{11,1} = \frac{1}{4}R_\Phi \left( e_1 \left( \partial_z^{-1}(\bar{b}q_2) - \sum_{k=1}^3 \tilde{m}_k(\bar{z}) \right) \right)$$

and we take  $v_{11,2}$  as

$$(3.45) \quad v_{11,2} = \frac{1}{4}e_2(x) \left( \partial_z^{-1}(\bar{b}q_2) - \sum_{k=1}^3 \tilde{m}_k(\bar{z}) \right) / \left( \tau \frac{\overline{\partial\Phi}}{\partial z} \right).$$

Thanks to our assumption on the function  $e_2$ , this function is smooth. Let us show that we can apply Proposition 3.4 to the function  $v_{11,1}$ . In order to do that we need to represent the function

$$(3.46) \quad \mathcal{G}_2 = e_1 \left( \partial_z^{-1}(\bar{b}q_2) - \sum_{k=1}^3 \tilde{m}_k(\bar{z}) \right)$$

in the form

$$\mathcal{G}_2 = zg(x),$$

where  $g$  is some function from  $C^1(\bar{\Omega})$ . This is an equivalent representation of the function  $m = \partial_z^{-1}(\bar{b}q_2) - \sum_{k=1}^3 \tilde{m}_k(\bar{z})$  in the form

$$m = r(z)g_1, \quad g_1 \in C^1(\bar{\Omega}).$$

Denote as  $p = \partial_z^{-1}(\bar{b}q_2)$ . Let  $x_j$  be a critical point of the function  $Im\Phi$  and  $z_j$  be an arbitrary critical point of the function  $\Phi$ . By Taylor's formula  $p(x) = p(x_j) + p_1(x_j)(z - z_j) + p_2(x_j)(\bar{z} - \bar{z}_j) + p_{11}(z - z_j)^2 + p_{12}(z - z_j)(\bar{z} - \bar{z}_j) + p_{22}(\bar{z} - \bar{z}_j)^2 + q(z, \bar{z})$ . Then  $m = p_1(x_j)(z - z_j) + p_{11}(z - z_j)^2 + p_{12}(z - z_j)(\bar{z} - \bar{z}_j) + q(z, \bar{z})$  and we set  $g_1 = (p_1(x_j)(z - z_j) + p_{11}(z - z_j)^2 + p_{12}(z - z_j)(\bar{z} - \bar{z}_j) + q(z, \bar{z}))/r(z)$ . Let us show that  $g_1 \in C^1(\bar{\Omega})$ . Obviously  $(p_1(z - z_j) + p_{11}(z - z_j)^2 + p_{12}(z - z_j)(\bar{z} - \bar{z}_j))/r(z)$  is a smooth function and  $\tilde{q}(z, \bar{z}) = q(z, \bar{z})/r(z)$  is  $C^1$  outside of  $z = 0$ . Continue the function  $\tilde{q}$  by zero on  $z = 0$ . Since  $q = o(|z|^3)$  the partial derivatives of this function vanishes at zero.

By Proposition 3.4

$$(3.47) \quad \|v_{11,2}\|_{L^2(\Omega)} + \|v_{11,1}\|_{L^2(\Omega)} \leq C(\delta)/\tau^{1-\delta}, \quad \forall \delta \in (0, 1).$$

Let  $v_{12}$  be a solution to the problem

$$(3.48) \quad \Delta(v_{12}e^{-\tau\varphi_1}) + q_2v_{12}e^{-\tau\varphi_1} = (q_2v_{11} + \Delta v_{11,2})e^{-\tau\Phi} - L_2 \left( \chi_2 e^{-\tau\Phi(\frac{1}{\bar{z}})} \overline{b \left( \frac{1}{\bar{z}} \right)} \right) \quad \text{in } \Omega,$$

$$(3.49) \quad v_{12}|_S = v_{11}e^{\tau Im\Phi}.$$

Then since

$$\|q_2v_{11} + \Delta v_{11,2}\|_{L^2(\Omega)} \leq C(\delta)/\tau^{1-\delta}, \quad \forall \delta \in (0, 1)$$

and by (3.41)

$$\left\| L_2 \left( \chi_2 e^{-\tau \overline{\Phi(\frac{1}{z})}} \overline{b \left( \frac{1}{z} \right)} \right) e^{\tau \varphi_1} \right\|_{L^2(\Omega)} = o \left( \frac{1}{\tau^2} \right),$$

and by (3.16), (3.45), (3.44)

$$\|v_{11}\|_{C^0(\partial\Omega)} \leq \frac{C}{\tau},$$

by Proposition 2.2 there exists a solution to problem (3.48) such that

$$(3.50) \quad \|v_{12}\|_{L^2(\Omega)} \leq C(\delta)/\tau^{\frac{3}{4}-\delta}, \quad \forall \delta \in (0, 1).$$

#### 4. Proof of the theorem

**Proposition 4.1.** *Suppose that  $\Phi$  satisfies (2.1), (2.2), (3.5) and (3.6). Let  $\{x_1, \dots, x_\ell\}$  be the set of critical points of the function  $\text{Im}\Phi$ . Then for any potentials  $q_1, q_2 \in C^\ell(\overline{\Omega})$ ,  $\ell > 1$  with the same DN maps and for any holomorphic functions  $a$  and  $b$ , we have*

$$\sum_{k=1}^{\ell} \frac{(q a \bar{b})(x_k)}{|\det \text{Im}\Phi''(x_k)|^{\frac{1}{2}}} = 0, \quad q = q_1 - q_2.$$

*Proof.* Let  $u_1$  be a solution to (3.3) and satisfy (3.38), and  $u_2$  be a solution to the following equation

$$\Delta u_2 + q_2 u_2 = 0 \quad \text{in } \Omega, \quad u_2|_{\partial\Omega} = u_1, \quad \nabla u_2|_{\Gamma_{-\epsilon}} = \nabla u_1.$$

Denoting  $u = u_1 - u_2$  we obtain

$$(4.1) \quad \Delta u + q_2 u = -q u_1 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = \frac{\partial u}{\partial \nu}|_{\Gamma_{-\epsilon}} = 0.$$

We multiply (4.1) by  $v$  and integrate over  $\Omega$ . By (3.36) and (3.50), we have

$$(4.2) \quad \begin{aligned} 0 &= \int_{\Omega} q u_1 v dx = \int_{\Omega} q (a \bar{b} + \bar{b} u_{11} + a v_{11}) e^{\tau(\Phi(z) - \overline{\Phi(z)})} dx \\ &+ \int_{\Omega} \left( q \chi_1(x) e^{\tau \Phi(\frac{1}{z})} a \left( \frac{1}{z} \right) \bar{b} e^{-\tau \overline{\Phi(z)}} + q \chi_1(x) e^{-\tau \overline{\Phi(\frac{1}{z})}} \overline{b \left( \frac{1}{z} \right)} a e^{\tau \Phi(z)} \right. \\ &\left. + q \chi_1(x) e^{\tau \Phi(\frac{1}{z})} a \left( \frac{1}{z} \right) \chi_2(x) e^{-\tau \overline{\Phi(\frac{1}{z})}} \overline{b \left( \frac{1}{z} \right)} \right) dx + o \left( \frac{1}{\tau} \right). \end{aligned}$$

By Propositions 3.1 and 3.6

$$\int_{\Omega} \left( q \chi_1(x) e^{\tau \Phi(\frac{1}{z})} a \left( \frac{1}{z} \right) \bar{b} e^{-\tau \overline{\Phi(z)}} + q \chi_2(x) e^{-\tau \overline{\Phi(\frac{1}{z})}} \overline{b \left( \frac{1}{z} \right)} a e^{\tau \Phi(z)} \right) dx = o \left( \frac{1}{\tau} \right).$$

By (3.42)

$$\int_{\Omega} q \chi_1(x) e^{\tau \Phi(\frac{1}{z})} a \left( \frac{1}{z} \right) \chi_2(x) e^{-\tau \overline{\Phi(\frac{1}{z})}} \overline{b \left( \frac{1}{z} \right)} dx = 0.$$

Therefore we can rewrite (4.2) as

$$(4.3) \quad \sum_{k=1}^{\ell} \frac{\pi (q a \bar{b})(x_k) e^{2i\tau \text{Im}\Phi(x_k)}}{\tau |\det \text{Im}\Phi''(x_k)|^{\frac{1}{2}}} + \int_{\Omega} q (\bar{b} u_{11} + a v_{11}) e^{\tau(\Phi(z) - \overline{\Phi(z)})} dx + o \left( \frac{1}{\tau} \right) = 0.$$

By (3.31), (3.45), (3.28), (3.43) and the fact that

$$(4.4) \quad \int_{\Omega} \bar{b}q u_{11,2} e^{\tau(\Phi(z) - \overline{\Phi(z)})} dx = \frac{1}{4\tau} \int_{\Omega} \bar{b}q \frac{e_2(\partial_{\bar{z}}^{-1}(aq_1) - \sum_{k=1}^3 m_k(z))}{\tau \partial_z \Phi(z)} e^{\tau(\Phi(z) - \overline{\Phi(z)})} dx = o\left(\frac{1}{\tau}\right),$$

and the fact that

$$(4.5) \quad \int_{\Omega} aqv_{11,2} e^{\tau(\Phi(z) - \overline{\Phi(z)})} dx = \frac{1}{4\tau} \int_{\Omega} aq \frac{e_2(\partial_z^{-1}(\bar{b}q_2) - \sum_{k=1}^3 \tilde{m}_k(\bar{z}))}{\tau \partial_z \Phi(z)} e^{\tau(\Phi(z) - \overline{\Phi(z)})} dx = o\left(\frac{1}{\tau}\right),$$

which follows from the stationary phase argument and  $e_2|_{\mathcal{H}} = 0$ , we obtain

$$(4.6) \quad \sum_{k=1}^{\ell} \frac{\pi(qa\bar{b})(x_k) e^{2i\tau \operatorname{Im} \Phi(x_k)}}{\tau |(\det \operatorname{Im} \Phi'')(x_k)|^{\frac{1}{2}}} + \int_{\Omega} q(\bar{b}u_{11,1} + av_{11,1}) e^{\tau(\Phi(z) - \overline{\Phi(z)})} dx + o\left(\frac{1}{\tau}\right) = 0.$$

By (3.13), (3.45) and (3.30)

$$(4.7) \quad \begin{aligned} 0 &= \sum_{k=1}^{\ell} \frac{\pi(qa\bar{b})(x_k) e^{2i\tau \operatorname{Im} \Phi(x_k)}}{\tau |(\det \operatorname{Im} \Phi'')(x_k)|^{\frac{1}{2}}} - \frac{1}{4} \int_{\Omega} q(\bar{b}\tilde{R}_{\Phi} \mathcal{G}_1 + aR_{\Phi} \mathcal{G}_2) e^{\tau(\Phi(z) - \overline{\Phi(z)})} dx + o\left(\frac{1}{\tau}\right) = \\ & \sum_{k=1}^{\ell} \frac{\pi(qa\bar{b})(x_k) e^{2i\tau \operatorname{Im} \Phi(x_k)}}{\tau |(\det \operatorname{Im} \Phi'')(x_k)|^{\frac{1}{2}}} - \frac{1}{4} \int_{\Omega} ((\partial_z^{-1}(q\bar{b}))\mathcal{G}_1 + (\partial_z^{-1}(qa))\mathcal{G}_2) e^{\tau(\Phi(z) - \overline{\Phi(z)})} dx + o\left(\frac{1}{\tau}\right) = \\ & \sum_{k=1}^{\ell} \frac{\pi(qa\bar{b})(x_k) e^{2i\tau \operatorname{Im} \Phi(x_k)}}{\tau |(\det \operatorname{Im} \Phi'')(x_k)|^{\frac{1}{2}}} + o\left(\frac{1}{\tau}\right). \end{aligned}$$

We remind the definitions of the functions  $\mathcal{G}_1$  and  $\mathcal{G}_2$  introduced in (3.32) and (3.46).

In order to get rid of the integral  $\int_{\Omega} ((\partial_z^{-1}(q\bar{b}))\mathcal{G}_1 + (\partial_z^{-1}(qa))\mathcal{G}_2) e^{\tau(\Phi(z) - \overline{\Phi(z)})} dx$ , we used the stationary phase lemma (see e.g. Theorem 7.7.5 [14]) and the fact that  $\mathcal{G}_1|_{\mathcal{H}} = \mathcal{G}_2|_{\mathcal{H}} = 0$ . Passing to the limit in this equality as  $\tau \rightarrow +\infty$  we obtain

$$\lim_{\tau \rightarrow \infty} \sum_{k=1}^{\ell} \frac{\pi(qa\bar{b})(x_k) e^{2i\tau \operatorname{Im} \Phi(x_k)}}{|(\det \operatorname{Im} \Phi'')(x_k)|^{\frac{1}{2}}} = 0.$$

The function  $K(\tau) \equiv \sum_{k=1}^{\ell} \frac{2\pi(qa\bar{b})(x_k) e^{2i\tau \operatorname{Im} \Phi(x_k)}}{|(\det \operatorname{Im} \Phi'')(x_k)|^{\frac{1}{2}}}$  is almost periodic. Therefore by the Bohr theorem (e.g., [3], p.493), we see that  $K(\tau) = 0$  for all  $\tau \in \mathbf{R}$ . Thus setting  $\tau = 0$ , we complete the proof.  $\square$

Proposition 4.1 plays the key role in the proof of Theorem 1.1. In order to be able to use this proposition we need to prove the existence of the weight function  $\Phi$ . The following proposition will allow us to construct this function.

Let  $\mathcal{P}_{\epsilon}$  be a non-empty open subset of the boundary  $\partial\Omega$ : the union of the segment between  $\hat{x}_+$  and  $\hat{x}_{+,\epsilon}$  and the segment between  $\hat{x}_{-,\epsilon}$  and  $\hat{x}_-$ .

Consider the Cauchy problem for the Laplace operator

$$(4.8) \quad \Delta\psi = 0 \quad \text{in } \Omega, \quad \left( \psi, \frac{\partial\psi}{\partial\nu} \right) \Big|_{\partial\Omega \setminus \mathcal{P}_\epsilon} = (a, b).$$

The following proposition establishes the solvability of (4.8) for a dense set of Cauchy data.

**Proposition 4.2.** *There exist a set  $\mathcal{O} \subset C^2(\overline{\partial\Omega \setminus \mathcal{P}_\epsilon}) \times C^1(\overline{\partial\Omega \setminus \mathcal{P}_\epsilon})$  such that for each  $(a, b) \in \mathcal{O}$ , problem (4.8) has at least one solution  $\psi \in C^2(\overline{\Omega})$  and  $\overline{\mathcal{O}} = C^2(\overline{\partial\Omega \setminus \mathcal{P}_\epsilon}) \times C^1(\overline{\partial\Omega \setminus \mathcal{P}_\epsilon})$ .*

*Proof.* First we observe that without the loss of generality we may assume that  $a \equiv 0$ . Consider the following extremal problem

$$(4.9) \quad J(\psi) = \left\| \frac{\partial\psi}{\partial\nu} - b \right\|_{H^2(\partial\Omega \setminus \mathcal{P}_\epsilon)}^2 + \epsilon \|\psi\|_{H^2(\partial\Omega)}^2 + \frac{1}{\epsilon} \|\Delta^2\psi\|_{L^2(\Omega)}^2 \rightarrow \inf,$$

$$(4.10) \quad \psi \in \mathcal{X}.$$

Here  $\mathcal{X} = \{\delta(x) \mid \delta \in H^2(\Omega), \Delta^2\delta \in L^2(\Omega), \Delta\delta|_{\partial\Omega} = \delta|_{\partial\Omega \setminus \mathcal{P}_\epsilon} = 0, \delta|_{\partial\Omega} \in H^2(\partial\Omega), \frac{\partial\delta}{\partial\nu} \in H^2(\partial\Omega \setminus \mathcal{P}_\epsilon)\}$ .

For each  $\epsilon > 0$  there exists a unique solution to (4.9) and (4.10), which we denote as  $\widehat{\psi}_\epsilon$ . By the Fermat theorem (see e.g., [1] p. 155) we have

$$J'(\widehat{\psi}_\epsilon)[\delta] = 0, \quad \forall \delta \in \mathcal{X}.$$

This equality can be written in the form

$$\left( \frac{\partial\widehat{\psi}_\epsilon}{\partial\nu} - b, \frac{\partial\delta}{\partial\nu} \right)_{H^2(\partial\Omega \setminus \mathcal{P}_\epsilon)} + \epsilon(\widehat{\psi}_\epsilon, \delta)_{H^2(\partial\Omega)} + \frac{1}{\epsilon}(\Delta^2\widehat{\psi}_\epsilon, \Delta^2\delta)_{L^2(\Omega)} = 0.$$

This equality implies that the sequence  $\{\frac{\partial\widehat{\psi}_\epsilon}{\partial\nu}\}$  is bounded in  $H^2(\partial\Omega \setminus \mathcal{P}_\epsilon)$ , the sequence  $\{\epsilon\widehat{\psi}_\epsilon\}$  converges to zero in  $H^2(\partial\Omega)$  and  $\{\frac{1}{\epsilon}\Delta^2\widehat{\psi}_\epsilon\}$  is bounded in  $L^2(\Omega)$ .

Therefore there exist  $q \in H^2(\partial\Omega \setminus \mathcal{P}_\epsilon)$  and  $p \in L^2(\Omega)$  such that

$$(4.11) \quad \frac{\partial\widehat{\psi}_{\epsilon_k}}{\partial\nu} - b \rightharpoonup q \quad \text{weakly in } H^2(\partial\Omega \setminus \mathcal{P}_\epsilon)$$

and

$$(4.12) \quad \left( q, \frac{\partial\delta}{\partial\nu} \right)_{H^2(\partial\Omega \setminus \mathcal{P}_\epsilon)} + (p, \Delta^2\delta)_{L^2(\Omega)} = 0 \quad \forall \delta \in \mathcal{X}.$$

Next we claim that

$$(4.13) \quad \Delta p = 0 \quad \text{in } \Omega$$

in the sense of distributions. Suppose that (4.13) is already proved. This implies

$$(p, \Delta^2\delta)_{L^2(\Omega)} = 0 \quad \forall \delta \in H^4(\Omega), \quad \Delta\delta|_{\partial\Omega} = \frac{\partial\Delta\delta}{\partial\nu} \Big|_{\partial\Omega} = 0.$$

This equality and (4.12) imply that

$$(4.14) \quad \left( q, \frac{\partial\delta}{\partial\nu} \right)_{H^2(\partial\Omega \setminus \mathcal{P}_\epsilon)} = 0 \quad \forall \delta \in H^4(\Omega), \quad \Delta\delta|_{\partial\Omega} = \frac{\partial\Delta\delta}{\partial\nu} \Big|_{\partial\Omega} = 0.$$

Then using the trace theorem, we conclude that  $q = 0$  and (4.11) implies that

$$\frac{\partial \widehat{\psi}_{\epsilon_k}}{\partial \nu} - b \rightharpoonup 0 \quad \text{weakly in } H^2(\partial\Omega \setminus \mathcal{P}_\epsilon).$$

By the Sobolev embedding theorem

$$\frac{\partial \widehat{\psi}_{\epsilon_k}}{\partial \nu} - b \rightarrow 0 \quad \text{in } C^1(\partial\Omega \setminus \mathcal{P}_\epsilon).$$

Therefore the sequence  $\{\widehat{\psi}_{\epsilon_k} - \widetilde{\psi}_{\epsilon_k}\}$ , with

$$\Delta \widetilde{\psi}_{\epsilon_k} = \Delta \widehat{\psi}_{\epsilon_k} \quad \text{in } \Omega, \quad \widetilde{\psi}_{\epsilon_k}|_{\partial\Omega} = 0$$

represents the desired approximation for solution of the Cauchy problem (4.8).

Now we prove (4.13). Let  $\tilde{x}$  be an arbitrary point in  $\Omega$  and let  $\tilde{\chi}$  be a smooth function such that it is zero in some neighborhood of  $\partial\Omega \setminus \mathcal{P}_\epsilon$  and the set  $\mathcal{B} = \{x \in \Omega | \tilde{\chi}(x) = 1\}$  contains an open connected subset  $\mathcal{F}$  such that  $\tilde{x} \in \mathcal{F}$  and  $\mathcal{P}_\epsilon \cap \overline{\mathcal{F}}$  is an open set in  $\partial\Omega$ . By (4.12)

$$0 = (p, \Delta^2(\tilde{\chi}\delta))_{L^2(\Omega)} = (\tilde{\chi}p, \Delta^2\delta)_{L^2(\Omega)} + (p, [\Delta^2, \tilde{\chi}]\delta)_{L^2(\Omega)}.$$

That is,

$$(4.15) \quad (\tilde{\chi}p, \Delta^2\delta)_{L^2(\Omega)} + ([\Delta^2, \tilde{\chi}]^*p, \delta)_{L^2(\Omega)} = 0 \quad \forall \delta \in \mathcal{X}.$$

This equality implies that  $\tilde{\chi}p \in H^1(\Omega)$ .

Next we take another smooth cut off function  $\tilde{\chi}_1$  such that  $\text{supp } \tilde{\chi}_1 \subset \mathcal{B}$ . A neighborhood of  $\tilde{x}$  belongs to  $\mathcal{B}_1 = \{x | \tilde{\chi}_1 = 1\}$ , the interior of  $\mathcal{B}_1$  is connected, and  $\text{Int } \mathcal{B}_1 \cap \mathcal{P}_\epsilon$  contains an open subset  $\mathcal{O}$  in  $\partial\Omega$ . Similarly to (4.16) we have

$$(4.16) \quad (\tilde{\chi}_1p, \Delta^2\delta)_{L^2(\Omega)} + ([\Delta^2, \tilde{\chi}_1]^*p, \delta)_{L^2(\Omega)} = 0.$$

This equality implies that  $\tilde{\chi}_1p \in H^2(\Omega)$ . Let  $\omega$  be a domain such that  $\omega \cap \Omega = \emptyset$ ,  $\partial\omega \cap \partial\Omega \subset \mathcal{O}$  contains a set open in  $\partial\Omega$ .

We extend  $p$  on  $\omega$  by zero. Then

$$(\Delta(\tilde{\chi}_1p), \Delta\delta)_{L^2(\Omega \cup \omega)} + ([\Delta^2, \tilde{\chi}_1]^*p, \delta)_{L^2(\Omega \cup \omega)} = 0.$$

Hence

$$\Delta^2(\tilde{\chi}_1p) = 0 \quad \text{in } \text{Int } \mathcal{B}_1 \cup \omega, \quad p|_\omega = 0.$$

By the Holmgren theorem  $\Delta(\tilde{\chi}_1p)|_{\text{Int } \mathcal{B}_1} = 0$ , that is,  $(\Delta p)(\tilde{x}) = 0$ .  $\square$

**Completion of the proof of Theorem 1.1.** It suffices to prove that  $q(0) = 0$ . We take  $\mathcal{P}_\epsilon$  in the previous proposition to be the union of the segment between  $\hat{x}_+$  and  $\hat{x}_{+, \epsilon}$  and the segment between  $\hat{x}_{-, \epsilon}$  and  $\hat{x}_-$ .

We will show that  $q_1(0) = q_2(0)$ . By obvious changes of the argument below we can prove that  $q_1(x) = q_2(x)$  for any point  $x \in \Omega$ .

Suppose that  $\psi(x)$  is a solution to (4.8) for some Cauchy data. Next, since  $\Omega$  is simply connected, we construct a function  $\varphi$  such that the function  $\Phi(z) = \varphi(x) + i\psi(x)$  is holomorphic

in  $\Omega$ . Consider the function  $\tilde{\Phi}(z) = z^2\Phi(z)$ . Observe that  $Im\tilde{\Phi} = (x_1^2 - x_2^2)\psi(x) + 2x_1x_2\varphi(x)$ . In particular by (4.8) and the Cauchy-Riemann equations, we have

$$Im\tilde{\Phi}|_{\partial\Omega \setminus \mathcal{P}_\epsilon} = (x_1^2 - x_2^2)a(x) + 2x_1x_2c(x), \quad \frac{\partial c(x)}{\partial \tau} = b(x).$$

Since we can choose  $a, b$  from a dense set in  $C^1(\overline{\partial\Omega \setminus \mathcal{P}_\epsilon})$  and the tangential derivatives of  $(x_1^2 - x_2^2)$  and  $x_1x_2$  are not equal zero simultaneously, we can choose  $a, b$  such that

$$(4.17) \quad \frac{\partial Im\tilde{\Phi}}{\partial \tau}|_{\bar{\Gamma}_-} = \frac{\partial Re\tilde{\Phi}}{\partial \nu}|_{\bar{\Gamma}_-} < 0, \quad \frac{\partial Im\tilde{\Phi}}{\partial \tau}|_{\partial\Omega \setminus \Gamma_{-, \epsilon}} = \frac{\partial Re\tilde{\Phi}}{\partial \nu}|_{\partial\Omega \setminus \Gamma_{-, \epsilon}} > 0.$$

Obviously the function  $\tilde{\Phi}$  has a critical point at zero. We may assume that  $\partial_z^2\tilde{\Phi}(0) \neq 0$ . Really if  $\Phi(0) \neq 0$  then  $\partial_z^2\tilde{\Phi}(0) = 2\Phi(0)$ . If  $\Phi(0) = 0$  we modify this function by adding a small real number:  $\tilde{\Phi}(z) + \tilde{\epsilon}$ . Obviously we will have (4.17).

A general function  $\tilde{\Phi}$  may have degenerate critical points. In order to avoid them, we approximate the function  $\tilde{\Phi}$  in  $C^1(\bar{\Omega})$  by a sequence of holomorphic functions  $\{\tilde{\Phi}_k\}_{k=1}^\infty$  such that

$$(4.18) \quad \tilde{\Phi}_k \rightarrow \tilde{\Phi} \quad \text{in } C^1(\bar{\Omega}), \quad \frac{\partial Re\tilde{\Phi}_k}{\partial \nu}|_{\bar{\Gamma}_-} < 0, \quad \frac{\partial Re\tilde{\Phi}_k}{\partial \nu}|_{\partial\Omega \setminus \Gamma_{-, \epsilon}} > 0,$$

$$(4.19) \quad \mathcal{H}_k = \{z | \partial_z \tilde{\Phi}_k(z) = 0\}, \quad card\mathcal{H}_k < \infty, \quad H_k \cap \partial\Omega = \{\emptyset\}, \quad \partial_z^2\tilde{\Phi}_k(z_\ell) \neq 0, \quad \forall z_\ell \in \mathcal{H}_k.$$

Let us show that such a sequence exists. For any  $\epsilon_1 \in (0, 1)$  we consider a function  $\tilde{\Phi}(z/(1 + \epsilon_1))$ . Obviously

$$\tilde{\Phi}(\cdot/(1 + \epsilon_1)) \rightarrow \tilde{\Phi} \quad \text{in } C^1(\bar{\Omega}), \quad \text{as } \epsilon_1 \rightarrow +0.$$

Each function  $\tilde{\Phi}(z/(1 + \epsilon_1))$  is holomorphic in  $B(0, 1 + \epsilon_1)$  and in  $B(0, 1)$  it can be approximated by a polynomial. Let  $\epsilon_1 \in (0, 1)$  be an arbitrary but fixed. Consider the sequence of such polynomials. Let  $p(z) = \sum_{k=0}^\kappa c_k z^k$  be a polynomial from this sequence. Consider the polynomial  $p'(z) = \sum_{k=1}^\kappa k c_k z^{k-1} = \prod_{k=1}^\ell (z - \hat{z}_k)^{s(k)}$ . Here we assume  $\hat{z}_j \neq \hat{z}_k$  for  $k \neq j$ . Let us construct an approximation of the polynomial  $p(z)$  by a sequence of polynomials of the order  $\kappa$ . We do the construction in the following way. First pick up all  $s(k)$  such that  $s(k) \geq 2$ . Denote the set of such indices as  $\mathcal{U}$ . Let  $\hat{k} \in \mathcal{U}$ . Consider the sequences  $\{\hat{z}_{k, \ell_1, \epsilon_2}\}, \dots, \{\hat{z}_{k, \ell_{s(\hat{k})}, \epsilon_2}\}$  such that

$$\begin{aligned} \hat{z}_{k, \ell_j, \epsilon_2} &\rightarrow \hat{z}_k \quad \text{as } \epsilon_2 \rightarrow +0, \quad \forall \ell_j \in \{\ell_1, \dots, \ell_{s(\hat{k})}\}, \\ \hat{z}_{k, \ell_j, \epsilon_2} &\neq \hat{z}_{k, \ell_{j_1}, \epsilon_2}, \quad 1 \leq k \leq \kappa, \quad \text{if } \ell_j \neq \ell_{j_1}. \end{aligned}$$

The polynomial

$$p'_{\epsilon_2}(z) = \prod_{k=1}^\ell \prod_{j=1}^{s(k)} (z - \hat{z}_{k, j, \epsilon_2})$$

does not have any zeros of order greater than one. By the construction we have

$$p'_{\epsilon_2}(z) = \sum_{k=1}^\kappa k c_{k, \epsilon_2} z^{k-1}$$

satisfying

$$c_{k, \epsilon_2} \rightarrow c_k, \quad \forall k \in \{1, \dots, \kappa\}.$$

This means that the sequence of polynomials  $p_{\epsilon_2}(z) = \sum_{k=0}^{\kappa} c_{k,\epsilon_2} z^k$ ,  $c_{0,\epsilon_2} = c_0$  converges to  $p(z)$  in  $C^1(\Omega)$  and for small  $\epsilon_2$  these polynomials do not have critical points.

Let us fix some sufficiently large  $\widehat{k}$  and consider  $k > \widehat{k}$ . Then  $\text{card } \mathcal{H}_{k_1} = \text{card } \mathcal{H}_{k_2}$  for all  $k_1 > \widehat{k}$  and  $k_2 > \widehat{k}$ . Let  $\text{card } \mathcal{H}_k = \ell$  and points  $z_1 = \tilde{x}_{1,1} + i\tilde{x}_{2,1}, \dots, z_\ell = \tilde{x}_{1,\ell} + i\tilde{x}_{2,\ell}$  represent all critical points of the function  $\tilde{\Phi}_k(z) = \varphi_k(z) + i\psi_k(z)$ .

Thanks to (4.18) and (4.19), we can apply Proposition 4.1. We have

$$\sum_{j=1}^{\ell} \frac{q(\tilde{x}_j)}{|\det \psi_k''(\tilde{x}_j)|^{\frac{1}{2}}} = 0, \quad \tilde{x}_j = (\tilde{x}_{1,j}, \tilde{x}_{2,j}).$$

Let  $\widehat{j} \in \{1, \dots, \ell\}$  be an arbitrary number. Consider the polynomial

$$p(z) = \frac{d_1}{2} \frac{\prod_{k \neq \widehat{j}}^{\ell} (z - z_k)^3}{\prod_{k \neq \widehat{j}}^{\ell} (z_j - z_k)^3} (z - z_j)^2 + d \frac{\prod_{k \neq \widehat{j}}^{\ell} (z - z_k)^3}{\prod_{k \neq \widehat{j}}^{\ell} (z_j - z_k)^3} (z - z_j).$$

Then

$$(4.20) \quad \begin{aligned} \partial_z^2 p(z_j) &= d_1 \in \mathbb{C}, \quad \partial_z p(z_j) = d \in \mathbb{C}, \\ p(z_j) &= \partial_z p(z_j) = \partial_z^2 p(z_j) = 0 \quad j \in \{1, \dots, \ell\} \setminus \{\widehat{j}\}. \end{aligned}$$

Consider the function  $\tilde{\Phi}_k(z) + \varepsilon p(z)$ . By (4.20) for small  $\varepsilon$  the set of critical points of this function consists exactly of  $\ell$  points, which we denote as  $z_j(\varepsilon)$  ( $\tilde{x}_j(\varepsilon) = (\text{Re} z_j(\varepsilon), \text{Im} z_j(\varepsilon))$ ). These critical points have the following properties:

$$(4.21) \quad z_j(0) = z_j, \quad \frac{\partial z_j(\varepsilon)}{\partial \varepsilon} \Big|_{\varepsilon=0} = 0, \quad j \neq \widehat{j}, \quad \frac{\partial z_{\widehat{j}}(\varepsilon)}{\partial \varepsilon} \Big|_{\varepsilon=0} = -\frac{d}{\partial_z^2 \tilde{\Phi}_k(z_{\widehat{j}})}.$$

In fact, there exists  $\varepsilon_0 > 0$  such that

$$z_j = z_j(\varepsilon), \quad \forall \varepsilon \in (-\varepsilon_0, \varepsilon_0), \quad j \neq \widehat{j}.$$

Then by Proposition 4.1 we have

$$J(\varepsilon) = \sum_{j=1}^{\ell} \frac{q(\tilde{x}_j(\varepsilon))}{|\det(\psi_k + \varepsilon \text{Im } p)''(\tilde{x}_j(\varepsilon))|^{\frac{1}{2}}} = 0.$$

Taking the derivative of the function  $J(\varepsilon)$  at zero, we have:

$$(4.22) \quad \begin{aligned} & -\frac{1}{|\partial_z^2 \tilde{\Phi}_k(z_{\widehat{j}})|^2} \frac{\partial_{x_1} q(\tilde{x}_{\widehat{j}}(0)) \overline{\text{Re}(d \partial_z^2 \tilde{\Phi}_k(z_{\widehat{j}}))} + \partial_{x_2} q(\tilde{x}_{\widehat{j}}(0)) \overline{\text{Im}(d \partial_z^2 \tilde{\Phi}_k(z_{\widehat{j}}))}}{|\det \psi_k''(\tilde{x}_{\widehat{j}}(0))|^{\frac{1}{2}}} \\ & + \frac{1}{2} \sum_{j=1}^{\ell} \left( \frac{q(\tilde{x}_j(0)) (2 \partial_{x_1 x_2}^2 \psi_k(\tilde{x}_j(0)) \text{Im } \partial_{x_1 x_2}^2 p(\tilde{x}_j(0)) + 2 \partial_{x_1 x_1}^2 \psi_k(\tilde{x}_j(0)) \text{Im } \partial_{x_1 x_1}^2 p(\tilde{x}_j(0)))}{|\det \psi_k''(\tilde{x}_j(0))|^{\frac{3}{2}}} \right. \\ & \left. + \frac{1}{2} q(\tilde{x}_j(0)) \frac{\partial_{x_1} (\det \psi_k''(\tilde{x}_j(0)) \overline{\text{Re}(d \partial_z^2 \tilde{\Phi}_k(z_{\widehat{j}})}) + \partial_{x_2} (\det \psi_k''(\tilde{x}_j(0)) \overline{\text{Im}(d \partial_z^2 \tilde{\Phi}_k(z_{\widehat{j}}))})}{|\partial_z^2 \tilde{\Phi}_k(z_{\widehat{j}})|^2 |\det \psi_k''(\tilde{x}_j(0))|^{\frac{3}{2}}} \right) = 0. \end{aligned}$$

The first and third terms of (4.22) are independent of  $\text{Im } \partial_{x_1 x_2}^2 p(\tilde{x}_j(0))$  and  $\text{Im } \partial_{x_1 x_1}^2 p(\tilde{x}_j(0))$ . Consequently

$$\frac{1}{2} \sum_{j=1}^{\ell} \frac{q(\tilde{x}_j(0))(-2\partial_{x_1 x_2}^2 \psi_k(\tilde{x}_j(0))\text{Im } \partial_{x_1 x_1}^2 p(\tilde{x}_j(0)) - 2\partial_{x_1 x_1}^2 \psi_k(x_j(0))\text{Im } \partial_{x_1 x_1}^2 p(\tilde{x}_j(0)))}{|\det \psi_k''(\tilde{x}_j(0))|^{\frac{3}{2}}} = 0.$$

This formula and (4.21) imply that  $q(\tilde{x}_j(0)) = 0$ . Since by (4.18) and (4.19) the set  $\mathcal{H}_k$  converges to the set of critical points of  $\tilde{\Phi}$  and 0 belongs to the set of critical points of  $\tilde{\Phi}$ , we have  $q(0) = 0$ . ■

## REFERENCES

- [1] V. Alekseev, V. Tikhomirov, S. Fomin, *Optimal Control*, Consultants Bureau, New York, 1987.
- [2] K. Astala, L. Päivärinta, *Calderón's inverse conductivity problem in the plane*, Ann. of Math., **163** (2006), 265–299.
- [3] A. Böttcher, B. Silbermann, *Analysis of Toeplitz Operators*, Springer-Verlag, Berlin, 2006.
- [4] R. Brown, R. Torres, *Uniqueness in the inverse conductivity problem for conductivities with 3/2 derivatives in  $L^p$ ,  $p > 2n$* , J. Fourier Analysis Appl., **9** (2003), 1049–1056.
- [5] R. Brown, G. Uhlmann, *Uniqueness in the inverse conductivity problem with less regular conductivities in two dimensions*, Comm. PDE., **22** (1997), 1009–10027.
- [6] A. Bukhgeim, *Recovering the potential from Cauchy data in two dimensions*, J. Inverse Ill-Posed Probl., **16** (2008), 19–34.
- [7] A. Bukhgeim, G. Uhlmann, *Recovering a potential from partial Cauchy data*, Comm. PDE., **27** (2002), 653–668.
- [8] A.P. Calderón, *On an inverse boundary value problem*, in *Seminar on Numerical Analysis and its Applications to Continuum Physics*, 65–73, Soc. Brasil. Mat., Rio de Janeiro, 1980.
- [9] J. Cheng, M. Yamamoto, *Determination of two convection coefficients from Dirichlet to Neumann map in the two-dimensional case*, SIAM J. Math. Anal., **35** (2004), 1371–1393.
- [10] D. Dos Santos Ferreira, C. E. Kenig, J. Sjöstrand, G. Uhlmann, *Determining a magnetic Schrödinger operator from partial Cauchy data*, Comm. Math. Phys., **271** (2007), 467–488.
- [11] D. Dos Santos Ferreira, C. Kenig, J. Sjöstrand, G. Uhlmann, *Limiting Carleman weights and anisotropic inverse problems*, preprint.
- [12] A. Greenleaf, M. Lassas, G. Uhlmann, *The Calderón problem for conormal potentials, I: Global uniqueness and reconstruction*, Comm. Pure Appl. Math., **56** (2003), 328–352.
- [13] H. Heck, J.-N. Wang, *Stability estimates for the inverse boundary value problem by partial Cauchy data*, Inverse Problems, **22** (2006), 1787–1796.
- [14] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer-Verlag, Berlin, 1985.
- [15] H. Kang, G. Uhlmann, *Inverse problems for the Pauli Hamiltonian in two dimensions*, Journal of Fourier Analysis and Applications, **10** (2004), 201–215.
- [16] C. Kenig, J. Sjöstrand, G. Uhlmann, *The Calderón problem with partial data*, Ann. of Math., **165** (2007), 567–591.
- [17] K. Knudsen, *The Calderón problem with partial data for less smooth conductivities*, Comm. PDE., **31** (2006), 57–71.
- [18] K. Knudsen, M. Salo, *Determining non-smooth first order terms from partial boundary measurements*, Inverse Problems and Imaging, **1** (2007), 349–369.
- [19] A. Nachman, *Global uniqueness for a two-dimensional inverse boundary value problem*, Ann. of Math., **143** (1996), 71–96.
- [20] L. Päivärinta, A. Panchenko, G. Uhlmann, *Complex geometrical optics for Lipschitz conductivities*, Revista Matemática Iberoamericana, **19** (2003), 57–72.

- [21] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer-Verlag, Berlin, 1992.
- [22] J. Sylvester, G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math., **125** (1987), 153–169.
- [23] L. Tzou, *Stability estimates for coefficients of magnetic Schrödinger equation from full and partial measurements*, to appear *Comm. PDE*.
- [24] G. Uhlmann, *Commentary on Calderón's paper (29) "On an inverse boundary value problem"*, Selected Papers of A.P. Calderón, edited by Alexandra Bellow, Carlos Kenig and Paul Malliavin, AMS, (2008), 623–636.
- [25] G. Uhlmann, J.-N. Wang, *Complex geometrical optics solutions and reconstruction of discontinuities*, SIAM J. Appl. Math., **68** (2008), 1026–1044.
- [26] I. Vekua, *Generalized Analytic Functions*, Pergamon Press, Oxford, 1962.

DEPARTMENT OF MATHEMATICS, COLORADO STATE UNIVERSITY, 101 WEBER BUILDING, FORT COLLINS CO, 80523 USA, E-MAIL: OLEG@MATH.COLOSTATE.EDU

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WA 98195 USA, E-MAIL: GUNTHER@MATH.WASHINGTON.EDU

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TOKYO, KOMABA, MEGURO, TOKYO 153, JAPAN, E-MAIL: MYAMA@MS.U-TOKYO.AC.JP