

1. (a) Let

$$A = \begin{bmatrix} 5 & 1 & 1 \\ 0 & 6 & 2 \\ 0 & 0 & 5 \end{bmatrix}.$$

Find the eigenvalues of A , and for each eigenvalue find a basis of the corresponding eigenspace. Is A diagonalisable? If so, find an invertible matrix S and a diagonal matrix D with $AS = SD$.

- (b) Let

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 0 & 6 & 2 \\ 0 & 0 & 5 \end{bmatrix}.$$

Find the eigenvalues of A , and for each eigenvalue find a basis of the corresponding eigenspace. Is A diagonalisable? If so, find an invertible matrix S and a diagonal matrix D with $AS = SD$.

2. Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 2 & 2 & 1 \end{bmatrix}.$$

- (a) Find $\det(A)$.
(b) Compute A^{-1} or show that it does not exist.
(c) Let

$$V_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, V_2 = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, V_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, U = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}.$$

Find scalars c_1, c_2, c_3 with $c_1V_1 + c_2V_2 + c_3V_3 = U$.

- (d) B is a 3×3 matrix with $BV_1 = 1V_1, BV_2 = 0V_2, BV_3 = 2V_3$. Compute B^3U .

3. Let A be as in Question 2, and suppose C is a 3×3 matrix with $C^5A = T$, where

$$T = \begin{bmatrix} 4 & 3 & 7 \\ 0 & 8 & 1 \\ 0 & 0 & 5 \end{bmatrix}.$$

- (a) Find $\det(C)$.
(b) Find the number of solutions X to the matrix equation

$$CX = \begin{bmatrix} 2 \\ 9 \\ 7 \end{bmatrix}.$$

4. Let

$$Y = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}.$$

- (a) Find a 3×4 matrix A with rank 2 such that the matrix equation $AX = Y$ has at least one solution, or show that no such matrix exists.
 - (b) Find a 4×3 matrix B with rank 3 such that the matrix equation $BX = Y$ has no solutions, or show that no such matrix exists.
 - (c) Find a 4×3 matrix C whose range is \mathbb{R}^4 , or show that no such matrix exists.
 - (d) Find a 4×3 matrix G such that the matrix equation $GX = 0$ has exactly one solution, or show that no such matrix exists.
5. Suppose A is a 4×4 matrix such that 5 is an eigenvalue of A with algebraic multiplicity 1, and -5 is not an eigenvalue of A . Find $\text{rank}(A - 5I)$ and $\text{rank}(A + 5I)$.