508A Example Sheet 1

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(1) Let X and Y be smooth projective curves over $k = \mathbb{C}$ and $f\colon X \to Y$ a morphism. Show the following:

(a) $g(X) \geq g(Y)$.

(b) If $g(X) = g(Y)$, then either f is an isomorphism, or $g(Y) = 1$ and $f\colon X \to Y$ is unramified, or $g(Y) = 0$. Give some examples (hint: if $g(Y) = 1$ then Y is a complex torus $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ for some $\tau \in \mathbb{C} \setminus \mathbb{R}$, if $g(Y) = 0$ then $Y \simeq \mathbb{P}^1$).

(2) A smooth projective curve of genus 0 is isomorphic to $\mathbb{P}^1$ (proof later). Use this to prove Lüroth's theorem: Let $k = \mathbb{C}$. If K is a field such that $k \subset K \subset k(t)$, then $K \simeq k(s)$ (here s and t are indeterminates).

(3) Let $X = (F(X_0, X_1, X_2) = 0) \subset \mathbb{P}^2$ be a plane curve of degree d. If X is smooth, show that X admits a morphism to $\mathbb{P}^1$ of degree $d - 1$. If X has a node and no other singularities then X admits a morphism to $\mathbb{P}^1$ of degree $d - 2$. Here, a node is a singularity $P \in X$ such that X has two smooth branches at P which cross transversely. Equivalently we can choose affine coordinates x, y at $P \in \mathbb{P}^2$ such that locally $X = (f(x, y) = 0) \subset \mathbb{A}^2_{x,y}$ where $f(x, y) = xy + \cdots$ (and $\cdots$ denote higher order terms in x and y). (Hint: consider projections $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$).

(4) Let $k = \mathbb{C}$. Let $f \in k(t)$ be a rational function. Consider the map

$$F = (1 : f)\colon \mathbb{P}^1 \to \mathbb{P}^1.$$

Find the degree of F, the ramification points of F, and the ramification indices. (Hint: write $f = f_1/f_0$ where $f_0, f_1 \in k[t]$ are coprime. Show that the degree d of F equals $\max(\deg f_0, \deg f_1)$, and F is ramified at the zeroes of $f_1' f_0 - f_0' f_1$ and at ∞ iff $\deg f_1' f_0 - f_0' f_1 < 2d - 2$).