# 508A Example Sheet 2

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(1) Consider the action of $G = \mathbb{Z}/n\mathbb{Z}$ on $X = \mathbb{A}^2_{x,y}$ given by

$$(x, y) \mapsto (\zeta x, \zeta^{-1} y),$$

where $\zeta$ is a primitive $n$th root of unity. Find equations for the quotient $X/G$. Now do the same for $(x, y) \mapsto (\zeta x, \zeta y)$.

(2) (a) Show that the automorphism group of $\mathbb{P}^1$ is $\mathrm{PGL}(2, k)$.

(b) Suppose $G \subset \mathrm{PGL}(2, \mathbb{C})$ is a finite subgroup. Classify the possible ramification indices for the quotient map $q \colon \mathbb{P}^1 \to \mathbb{P}^1/G$ using the Riemann–Hurwitz formula. In each case give an example or explain why it cannot occur. (Hint: You should find cases corresponding to cyclic groups, dihedral groups, and the groups of rotations of the Platonic solids, under the homomorphism $\mathrm{SO}(3) \to \mathrm{PGL}(2, \mathbb{C})$ given by regarding $\mathbb{P}^1_{\mathbb{C}}$ as the unit sphere in $\mathbb{R}^3$. If you are stuck, see [Miranda, p. 80]).

(3) A complex torus (of dimension 1) is a quotient $X = \mathbb{C}/\Lambda$ where $\Lambda \subset \mathbb{C}$ is a lattice, that is, $\Lambda = \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2$ and $\lambda_2/\lambda_1 \notin \mathbb{R}$.

(a) Let $f \colon X \to X'$ be a morphism of complex tori $X = \mathbb{C}/\Lambda$, $X' = \mathbb{C}/\Lambda'$. Show that $f$ is of the form $z \mapsto az + b$, where $b \in \mathbb{C}$ is arbitrary and $a \in \mathbb{C}$ satisfies $a\Lambda \subseteq \Lambda'$. (Hint: Show that the derivative of $f$ gives a global holomorphic function on $\mathbb{C}/\Lambda$ and so is constant.)

(b) Use part (a) to describe the possible automorphism groups of complex tori of dimension 1.

(c) What about dimension $> 1$?

(4) Suppose that $e_i \in \mathbb{Z}$, $e_i \geq 1$, and $\sum(1-\frac{1}{e_i}) > 2$. Show that $\sum(1-\frac{1}{e_i}) \geq 2\frac{1}{42}$ with equality iff $\{e_i\} = \{2,3,7\}$. (Note: this fact was used in the proof of Hurwitz's bound for the number of automorphisms of a curve of genus $g \geq 2$).

(5) The Klein quartic is the curve $X = (X^3Y + Y^3Z + Z^3X = 0) \subset \mathbb{P}^2$.

   (a) Show that $X$ is smooth of genus 3.

   (b) Compute the automorphism group of $X$. (Hint: $\mathrm{Aut}(X)$ has the maximum possible order $84(g-1) = 168$. It is generated by 3 elements of orders 2,3,7 — the last two are fairly easy to spot. If you are stuck, try looking at [Thurston] or [Elkies]).

(6) Let $X = (X^d + Y^d + Z^d = 0) \subset \mathbb{P}^2$, the Fermat curve of degree $d$. Let $f\colon X \to \mathbb{P}^1$ be the degree $d$ morphism given by projection from the point $(0:0:1) \in \mathbb{P}^2$, that is, $(X:Y:Z) \mapsto (X:Y)$. Find the branch points of $f$ and describe the monodromy representation.

## References


[Elkies] N. Elkies, The Klein quartic in number theory, available at www.msri.org/publications/books/Book35/files/elkies.pdf

[Miranda] R. Miranda, Algebraic curves and Riemann surfaces.

[Thurston] W. Thurston, The eightfold way, available at www.msri.org/publications/books/Book35/files/thurston.pdf