508A Example Sheet 3

Paul Hacking

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(1) Let $X = (y^2 = x^3) \subset \mathbb{A}^2_{x,y}$, a singular plane curve. Let

$$A = k[X] = k[x, y]/(y^2 - x^3)$$

be the coordinate ring of X. Compute the module of Kähler differentials $\Omega_{A/k}$ using Lem. 2.11 from the lecture notes, part 3. Let K be the fraction field of A (the function field of X). Show that the natural map $\Omega_{A/k} \to \Omega_{K/k}$ is not injective. (This shows that we need to be careful when considering differential forms on singular varieties.)

(2) Let X be a smooth projective curve over $k = \mathbb{C}$. Recall (from the notes) the definition of the *Jacobian* $J(X)$ and the *Abel-Jacobi map*

$$A\colon \operatorname{Div}(X) \to J(X).$$

Here we prove that $A(D) = 0$ if $D = (f)$ is a principal divisor. We assume for simplicity that f has only simple zeroes and poles, and write $(f) = \sum_{i=1}^{d} P_i - \sum_{i=1}^{d} Q_i$

(a) Consider the degree d map $F = (1 : f)\colon X \to \mathbb{P}^1$. Let $\gamma\colon [0, 1] \to \mathbb{P}^1$ be a path from $(1 : 0)$ to $(0 : 1)$ whose interior does not contain any branch points of F. Show that γ lifts to d disjoint paths $\tilde{\gamma}_i\colon [0, 1] \to X$. Relabelling if necessary we may assume $\tilde{\gamma}_i(0) = P_i$, $\tilde{\gamma}_i(1) = Q_i$.

(b) Show that $\sum \int_{\tilde{\gamma}_i} \omega = \int_\gamma \operatorname{Tr}(\omega)$, where $\operatorname{Tr}(\omega)$ is the trace of ω on $\mathbb{P}^1$.

(c) Explain why $\operatorname{Tr}(\omega) = 0$.

(d) Deduce from (2) and (3) that $A(D) = 0$.

(3) Let $X = E$ be an elliptic curve (a curve of genus 1) over $k = \mathbb{C}$. Recall (from the lecture notes, part 4) that the associated complex manifold E^{an} is a complex torus $\mathbb{C}/L$. Here $L \subset \mathbb{C}$ is a lattice, that is, $L \simeq \mathbb{Z}^2$ and L spans $\mathbb{C}$ as an $\mathbb{R}$-vector space. Show that in this case the Abel-Jacobi map $A\colon E \to J(E)$ is an isomorphism. (Hint: First show that $\Omega_X(X) = \langle dz \rangle_{\mathbb{C}}$, where z is the coordinate on $\mathbb{C} \to \mathbb{C}/L$. Then explicitly compute the integrals in the definition of $J(E)$ and A).

(4) Let $X = \mathbb{P}^1$ and $D = nP$ where $P = (0:1) \in \mathbb{P}^1$.

(a) Show that

$$L(D) = \langle 1, \ldots, x^n \rangle$$

where $x = X_1/X_0$.

(b) Now consider the corresponding morphism

$$\phi_{|D|}\colon \mathbb{P}^1 \to \mathbb{P}^n, \quad x \mapsto (1, x, \ldots, x^n).$$

Show that $\phi_{|D|}$ is an embedding for $n \geq 1$ and its image Y is a curve of degree n in $\mathbb{P}^n$.

(c) Show that Y is defined by the vanishing of the 2×2 minors of the matrix

$$\begin{pmatrix} X_0 & X_1 & \cdots & X_{n-1} \\ X_1 & X_2 & \cdots & X_n \end{pmatrix}$$

where $X_0, \ldots, X_n$ are the homogeneous coordinates on $\mathbb{P}^n$. For example, if $n = 2$, then Y is the conic $(X_0X_2 = X_1^2) \subset \mathbb{P}^2$. If $n = 3$, Y is the *twisted cubic* in $\mathbb{P}^3$. In general, Y is called the *rational normal curve of degree n*.

(5) Let E be an elliptic curve over $k = \mathbb{C}$, and write $E^{\mathrm{an}} = \mathbb{C}/L$. Let $P \in E$ be the point $0 \in \mathbb{C}/L$. Recall that we constructed a meromorphic function $\wp = \wp(z) \in L(2P)$, the *Weierstrass $\wp$ function.* (Also, $L(P) = \mathbb{C}$.) Now let $D = nP$, $n \geq 2$.

(a) Show that

$$L(D) = \langle 1, \wp, \wp^{(1)}, \ldots, \wp^{(n-2)} \rangle_{\mathbb{C}}$$

where $\wp^{(i)}$ denotes the ith derivative of $\wp$ with respect to z.

(b) Now consider the corresponding morphism

$$\phi_{|D|}\colon E \to \mathbb{P}^n.$$

Show that $\phi_{|D|}$ is a degree 2 map to $\mathbb{P}^1$ for $n = 2$ and an embedding for $n \geq 3$, with image a curve of degree n.

(c) Show that $\phi_{|3P|}(E) \subset \mathbb{P}^2$ is given by an equation of the form

$$X_0 X_2^2 = X_1^3 + aX_0^2 X_1 + bX_0^3$$

where $a, b \in \mathbb{C}$. (Hint: Use the basis $1, \wp, \wp'$ of $L(3P)$. Show that $P \mapsto (0:0:1)$, and the line $(X_0 = 0)$ is tangent to E at P with order of contact 3. So the equation has the form $X_0 Q + X_1^3 = 0$ for some Q of degree 2. Now a careful change of coordinates gives the normal form above).

(6) Let $f\colon X \to \mathbb{P}^1$ be a hyperelliptic curve of genus g over $k = \mathbb{C}$. Let $Q \in \mathbb{P}^1$ be a branch point of f and $D = f^*Q = 2P$ where $P = f^{-1}(Q)$. Change coordinates on $\mathbb{P}^1$ so that $Q = (0:1)$ and let $x = X_1/X_0$.

(a) Let $n \leq g$. Show that

$$L(nD) = \langle 1, \ldots, x^n \rangle_k.$$

Describe the corresponding map $\phi_{|nD|}$.

(b) Show that if $n > g$ then $\phi_{|nD|}$ is an embedding.

Hint: X is covered by 2 affine pieces $U_0 = (y^2 = f(x)) \subset \mathbb{A}^2_{x,y}$ and $U_1 = (w^2 = g(z)) \subset \mathbb{A}^2_{z,w}$, with glueing

$$U_0 \supset (x \neq 0) \xrightarrow{\sim} (z \neq 0) \subset U_1, \quad (x, y) \mapsto (z, w) = (x^{-1}, x^{-(g+1)} y).$$

Here $f(x) = \prod_{i=0}^{2g+1}(x - \alpha_i)$ and $g(z) = z^{2g+2} f(z^{-1}) = z \prod_{i=1}^{2g-1}(1 - \alpha_i z)$ (recall that f is ramified over $(0:1)$).