Math 425, Winter 2018, Homework 1 Solutions

Pugh, Ch. 3: 19

(a.) We show that the complement of D_k is relatively open, that is, if $\mathrm{osc}_x(f) < \frac{1}{k}$ then there is some $r > 0$ such that if $y \in (x-r, x+r) \cap [a,b]$ then $\mathrm{osc}_y(f) < \frac{1}{k}$.

Consider first $x \in (a,b)$. Then since $\mathrm{osc}_x(f) < \frac{1}{k}$ we can find $r > 0$ so $(x-r, x+r) \subset [a,b]$ and

$$\sup_{t \in (x-r,x+r)} f(t) - \inf_{t \in (x-r,x+r)} f(t) < \frac{1}{k}.$$

If $y \in (x-r, x+r)$, then there is some $\delta > 0$ so $(y-\delta, y+\delta) \subset (x-r, x+r)$, and thus

$$\sup_{t \in (y-\delta,y+\delta)} f(t) - \inf_{t \in (y-\delta,y+\delta)} f(t) < \frac{1}{k} \quad \Rightarrow \quad \mathrm{osc}_y(f) < \frac{1}{k}.$$

If $x = a$ or $x = b$ the same proof works; for $x = a$ we have some $r > 0$ so

$$\sup_{t \in [a,a+r)} f(t) - \inf_{t \in [a,a+r)} f(t) < \frac{1}{k}.$$

We can consider $y \in (a, a+r)$ (since we know $\mathrm{osc}_y(f) < \frac{1}{k}$ for $y = a$), and then there is some $\delta > 0$ so $(y-\delta, y+\delta) \subset (a, a+r)$, and continue as above. Similarly for $x = b$.

(b.) The set of discontinuities equals $\bigcup_{k=1}^{\infty} D_k$, so is a countable union of closed sets.

(c.) The set of points of continuity equals $[a,b] \setminus \bigcup_{k=1}^{\infty} D_k = \bigcap_{k=1}^{\infty} [a,b] \setminus D_k$ is the countable intersection of (relatively) open sets.

Pugh, Ch. 3: 27(b)

An example is $\chi_{\mathbb{Q}}(x)$ over the set $[0,1]$. Each $x_k^* = \frac{1}{n}(k - \frac{1}{2})$ in the midpoint rule with n sample points is a rational sumber, so the midpoint rule leads to Riemann sums equal to 1 for every n. But as shown in class the function $\chi_{\mathbb{Q}}$ is not Riemann/Darboux integrable on $[0,1]$ (or any open interval for that matter). Note that if we had taken the integral over $[0,b]$ for b irrational then the midpoint Riemann sums would be identically 0.

Pugh, Ch. 3: 28(i$\Leftrightarrow$ii)

(i$\Rightarrow$ii): we can cover Z by countable intervals (a_j, b_j) with $\sum_{j=1}^{\infty} b_j - a_j < \epsilon$, and the sets $[a_j, b_j]$ give a cover satisfying (ii).

(i$\Leftarrow$ii): cover Z by closed intervals $[a_j, b_j]$ with $\sum_{j=1}^{\infty} b_j - a_j < \frac{1}{2}\epsilon$. Then the open intervals $(a_j - \frac{\epsilon}{2^{j+2}}, b_j + \frac{\epsilon}{2^{j+2}})$ cover, and

$$\sum_{j=1}^{\infty} b_j + \frac{\epsilon}{2^{j+2}} - \left(a_j - \frac{\epsilon}{2^{j+2}}\right) = \sum_{j=1}^{\infty} b_j - a_j + \epsilon \sum_{j=1}^{\infty} \frac{1}{2^{j+1}} < \epsilon.$$

Additional Problem 1:

Let x be irrational. Given $k \in \mathbb{N}$ we find $\delta > 0$ so $|y - x| < \delta \Rightarrow f(y) < \frac{1}{k}$. (Since $f(y) \geq 0$ for all y and $f(x) = 0$ this is the continuity criteria.)
There are only finitely many point $x_j = \frac{j}{k}$ with $0 \leq j \leq k$ for which $f(x_j) \geq \frac{1}{k}$. So let $\delta = \min_{0 \leq j \leq k} |y - x_j|$; then $\delta > 0$ since $x \notin \mathbb{Q}$, and $|y - x| < \delta \Rightarrow y \neq x_j$ for any j.

Additional Problem 2:

Recall we have a disjoint decomposition $\mathbb{R} = \partial S \cup \mathrm{int}(S) \cup \mathrm{int}(S^c)$. Suppose $x \in \partial S$. Then for all $r > 0$ there is some $y \in S$ such that $|y - x| < r$, and some other $y \in \mathbb{R} \backslash S$ such that $|y - x| < r$. Thus

$$\sup_{|y-x|<r} \chi_S(y) = 1 \text{ and } \inf_{|y-x|<r} \chi_S(y) = 0 \quad \Rightarrow \quad \mathrm{osc}_x(\chi_S) = 1.$$

On the other hand, if $x \in \mathrm{int}(S)$ then there is $r > 0$ so $|y - x| < r \Rightarrow y \in S \Rightarrow \chi_S(y) = 1$, and $x \in \mathrm{int}(S^c)$ then there is $r > 0$ so $|y - x| < r \Rightarrow y \in S^c \Rightarrow \chi_S(y) = 0$. This shows $\mathrm{osc}_x(\chi_S) = 0$ if $y \notin \partial S$.