Math 425, Winter 2018, Homework 2 Solutions

Pugh, Ch. 3: 51

We know $g(x) - f(x) > 0$ for every x. Also, there must be some point x_0 where both f and g are continuous: this is because the set where either f or g are discontinuous is a set of measure 0, but $[a, b]$ is not a set of measure 0. (Pugh assumes $a < b$ without stating it explicitly.) Then $g(x_0) - f(x_0) > 0$ and is continuous at x_0, so for some $\delta > 0$ we know $g(x) - f(x) > \frac{1}{2}(g(x_0) - f(x_0))$ if $|x - x_0| < \delta$. Since $g(x_0) - f(x_0) > 0$ for all x, we deduce

$$g(x) - f(x) > \frac{1}{2}\Big(g(x_0) - f(x_0)\Big) \cdot \chi_{(x_0-\delta, x_0+\delta)}(x) \qquad \forall x \in [a, b].$$

Thus

$$\int_a^b g(x)\,dx - \int_a^b f(x)\,dx > \frac{1}{2}\Big(g(x_0) - f(x_0)\Big) \int_a^b \chi_{(x_0-\delta, x_0+\delta)}(x)\,dx > 0,$$

where the last inequality holds since $(a, b) \cap (x_0 - \delta, x_0 + \delta)$ is a nonempty open interval.

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The proof follows by showing that the function $\max\big(f(x), g(x)\big)$ is continuous at a point x_0 if both f and g are continuous there, and similarly for $\min\big(f(x), g(x)\big)$. If will follow that the set of discontinuitites of $\max\big(f(x), g(x)\big)$ is measure 0.

There are a few ways of showing this; for example using sequences or an $\epsilon - \delta$ argument. Alternatively, one can write

$$\max\big(f(x), g(x)\big) = \frac{1}{2}\Big(|f(x) + g(x)| - |f(x) - g(x)|\Big)$$

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We are assuming that $a_k \geq 0$, so convergence of $\sum a_k$ is equivalent to existence of M such that, for all $m \in \mathbb{N}$,

$$\sum_{k=1}^{m} a_k \leq M.$$

This is equivalent to $\displaystyle\sum_{k=1}^{2^n} a_k \leq M$ holding for all n (since the a_k are nonnegative). We will prove

$$\frac{1}{2}\sum_{j=1}^{n} 2^j a_{2^j} \leq \sum_{k=1}^{2^n-1} a_k \leq \sum_{j=0}^{n-1} 2^j a_{2^j}. \tag{1}$$

It will then follow that $\sum_k a_k$ converges iff $\sum_j 2^j\, a_{2^j}$ converges.

It is easy to see (1) symbolically: since a_j is decreasing,

$$a_2 + 2a_4 + 4a_8 + \cdots \leq a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots \leq a_1 + 2a_2 + 4a_4 + \cdots$$

To prove (1) explicitly, we write $\displaystyle\sum_{k=1}^{2^n-1} a_k = \sum_{j=1}^{n} \sum_{k=2^{j-1}}^{2^j-1} a_k$, and since the sequence is decreasing

$$2^{j-1} a_{2^j} \leq \sum_{k=2^{j-1}}^{2^j-1} a_k \leq 2^{j-1} a_{2^{j-1}}.$$

Additional Problem 1:

We assume $f'(x)$ is Darboux integrable. Let $P = \{a = x_0 < \cdots < x_n = b\}$ be any partition of $[a, b]$, and recall

$$U(f', P) = \sum_{j=1}^{n} M_j(x_j - x_{j-1}), \quad L(f', P) = \sum_{j=1}^{n} m_j(x_j - x_{j-1}),$$

where

$$M_j = \sup_{t \in [x_{j-1}, x_j]} f'(t), \qquad m_j = \inf_{t \in [x_{j-1}, x_j]} f'(t).$$

If $j = 0$ or $j = n$ take the interval to be respectively left or right open; in fact the proof below will work if every interval is taken to be open at both ends when defining M_j and m_j.

By the mean value theorem, $f(x_j) - f(x_{j-1}) = f'(t)(x_j - x_{j-1})$ for some $t \in (x_{j-1}, x_j)$, so

$$m_j(x_j - x_{j-1}) \leq f(x_j) - f(x_{j-1}) \leq M_j(x_j - x_{j-1}).$$

Adding up over j we obtain, for any partition,

$$L(f', P) \leq f(b) - f(a) \leq U(f', P)$$

Taking a limit over partitions so $U(f', P) - L(f', P) \to 0$ we conclude that $f(b) - f(a) = \int_a^b f'(t)\, dt$.

Additional Problem 2:

(a.) Suppose $k \geq 1$. The function $f(x) = x^k : [0, \infty) \to [0, \infty)$ is continuous, and strictly increasing: if $x < z$ then $x^k < z^k$. This can be seen by the mean value theorem, using that $f'(x) = kx^{k-1} > 0$ on $(0, \infty)$, or by multiplication properties of positive numbers. Also, if $x \geq 1$ then $f(x) \geq x$.

Suppose $0 < y < M$, and take $M \geq 1$. Then since $f(0) < y < f(M)$, the intermediate value theorem says there is some $x \in (0, M)$ so that $f(x) = y$. By strict increasingnessity of f, x is unique.

(b.) Given $\epsilon > 0$, and $y > 0$, we need to find N such that $1 - \epsilon < y^{1/k} < 1 + \epsilon$ when $k \geq N$. Since $f(x) = x^k$ is increasing, this is equivalent to

$$(1 - \epsilon)^k < y < (1 + \epsilon)^k \quad \text{if } \ k \geq N.$$

Since $1 - \epsilon < 0$, we know $\lim_{k \to \infty}(1 - \epsilon)^k = 0$, and since $1 + \epsilon > 0$, we know $\lim_{k \to \infty}(1 + \epsilon)^k = \infty$, in the sense that given any R there is some N so $(1 + \epsilon)^k > M$ for $k \geq N$. Since $y > 0$ we conclude that there is some N so the above holds.