Math 425, Winter 2018, Homework 3 Solutions

Pugh, Ch. 4: 1(a)

We say that $f_n \to f$ pointwise if for each point $x \in M$ we have $\lim_{n\to\infty} f_n(x) = f(x)$.
We say that $f_n \to f$ uniformly if given $\epsilon > 0$ there exists N such that when $n \geq N$ we have $d_N\big(f_n(x), f(x)\big) < \epsilon$ at every $x \in M$.

Pugh, Ch. 4: 2

The proof is almost identical to that of Theorem 1. Given $\epsilon > 0$ find n so $d_N(f_n(y), f(y)) < \epsilon/3$ for all $y \in M$. Given $x \in M$ there is $\delta > 0$ so that $d_N(f_n(x), f_n(y)) < \epsilon/3$ if $d_M(x, y) < \delta$. Then if $d_M(x, y) < \delta$

$$d_N(f(x), f(y)) \leq d_N(f(x), f_n(x)) + d_N(f_n(x), f_n(y)) + d_N(f_n(y), f(y)) < \epsilon,$$

so f is continuous from $M \to N$.

Pugh, Ch. 4: 3

(a.) As in the proof of Theorem 1, find n so $|f_n(x) - f(x)| < \epsilon/3$ for all x. Then by continuity of f_n at x_0 there is $\delta > 0$ so $|f_n(x) - f_n(x_0)| < \epsilon/3$ if $|x - x_0| < \delta$. Then if $|x - x_0| < \delta$ we have

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| < \epsilon$$

so f is continuous at x_0.

(b.) f need not be piecewise continous even if each f_n is. An example is to let

$$f(x) = \begin{cases} \frac{1}{j}\,, & x \in (\frac{1}{j+1}, \frac{1}{j}] \\ 0\,, & x = 0. \end{cases}$$

and

$$f_n(x) = \begin{cases} \frac{1}{j}\,, & x \in (\frac{1}{j+1}, \frac{1}{j}] \quad \text{if } j \leq n \\ 0\,, & x \in [0, \frac{1}{n+1}]. \end{cases}$$

Pugh, Ch. 4: 4(a)

This is just like problem 2, taking $M = N = \mathbb{R}$, and noting that the choice of δ for f_n is independent of x and y, so we get $|f(x) - f(y)| < \epsilon$ for all $|x - y| < \delta$.

Additional Problem 1:

(a.) Suppose that $f_n \to f$ uniformly in $C_b(\mathbb{R})$. Then, given $\epsilon > 0$ there is some n so $\sup_{x\in\mathbb{R}} |f(x) - f_n(x)| < \epsilon/2$. Since $f_n \in C_0(\mathbb{R})$ there is some R so that $|f(x)| < \epsilon/2$ if $|x| > R$. Then

$$|x| > R \;\Rightarrow\; |f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $f \in C_0(\mathbb{R})$, and this shows that $C_0(\mathbb{R})$ is (sequentially) closed in $C_b(\mathbb{R})$.

(b.) Given $\epsilon > 0$, we need find δ so that $|f(x) - f(y)| < \epsilon$ if $|x - y| < \delta$. Since $f \in C_0(\mathbb{R})$ there is R so that $|f(x)| < \epsilon/2$ if $|x| \geq R$. Since $[-R-1, R+1]$ is compact, the function f is uniformy continuous there, so there is $\delta > 0$ so that $|f(x) - f(y)| < \epsilon$ if $|x - y| < \delta$ and $x, y \in [-R-1, R+1]$. We may assume $\delta < 1$. Now assume $x, y \in \mathbb{R}$ and $|x - y| < \delta$. Then either both $x, y \in [-R-1, R+1]$, or both $|x|, |y| \geq R$. In the former case we have $|f(x) - f(y)| \leq \delta$, and in the latter $|f(x) - f(y)| \leq |f(x)| + |f(y)| \leq \epsilon$.

(c.) $f(x) = \sin(x^2)$ is in $C_b(\mathbb{R})$ but is not uniformly continous.

Additional Problem 2: We use Taylor's theorem (Pugh page 160) to deduce

$$\Big| f(x) - \sum_{k=0}^{m} \frac{f^{(k)}(0)}{k!}\, x^k \Big| \leq \frac{C\,|x|^{m+1}}{R^{m+1}} = C\Big(\frac{|x|}{R}\Big)^{m+1}.$$

If $|x| < R$ the right hand side goes to 0 as $m \to \infty$.