Math 425, Winter 2018, Homework 4 Solutions

Pugh, Ch. 4: 9.

The function f must be constant. Given $\epsilon > 0$, we will show that $|f(x) - f(0)| < \epsilon$ for every $x \in \mathbb{R}$, which proves f is constant. We assume pointwise equicontinuity of $f(nx)$ at $x = 0$. Let $\delta > 0$ be so that $|y| < \delta \Rightarrow |f(ny) - f(0)| < \epsilon$. Given $x \neq 0$, find n so $n\delta > |x|$, so $|n^{-1}x| < \delta$. Then $|f(x) - f(0)| = |f(n \cdot n^{-1}x) - f(0)| < \epsilon$.

Pugh, Ch. 4: 12.

We show that if the condition

$$|x - y| \leq \delta \Rightarrow |f(x) - f(y)| \leq \epsilon$$

holds for all $f \in \mathcal{F}$, and $(f_n) \subset \mathcal{F}$ converges pointwise to g, then

$$|x - y| \leq \delta \Rightarrow |g(x) - g(y)| \leq \epsilon$$

To see this, take any two points x, y with $|x - y| \leq \delta$, and write

$$|g(x) - g(y)| = \lim_{n\to\infty} |f_n(x) - f_n(y)| \leq \epsilon.$$

Pugh, Ch. 4: 13.

(a.) Yes. For example, use the argument from Arzela-Ascoli to produce a subsequence (g_n) that converges pointwise at each $x \in \mathbb{Q}$. Pointwise equicontinuity and pointwise boundedness implies uniform equicontinuity and boundedness on any compact interval, so the Arzela-Ascoli Theorem (or Theorem 16) shows that the sequence converges uniformly on each compact interval to a continuous function. This determines a function $g(x)$ for $x \in \mathbb{R}$, that is continuous on $\mathbb{R}$ since it is continuous on each interval $(-R, R)$ for every $R > 0$.

(b.) The convergence need not be uniform on $\mathbb{R}$. For example, $\big(1 + |x - n|^2\big)^{-1}$ is equicontinuous on $\mathbb{R}$, and converges pointwise to 0, but not uniformly to 0.

Pugh, Ch. 4: 15.

(a.) If f has modulus of continuity $\mu(s)$, given ϵ we can find δ so $\mu(\delta) < \epsilon$, since $\lim_{\delta\to 0} \mu(\delta) = 0$. Then $|x - y| < \delta$ implies

$$|f(x) - f(y)| < \mu(|x - y|) \leq \mu(\delta) < \epsilon$$

where we use that μ is increasing. (There is no need for strictly increasing here.)

For the other direction, assume f is uniformly continuous. Define

$$\mu(s) = \sup\big\{\, |f(x) - f(y)| : |x - y| \leq s\big\}.$$

Then $\mu(s)$ is increasing in s (though not strictly) since one is taking the sup of a larger set the larger s is. To see that $\lim_{s\to 0} \mu(s) = 0$, note that if $\epsilon > 0$ there is some $\delta > 0$ so that $|x - y| \leq \delta$ then $|f(x) - f(y)| \leq \epsilon$. This implies that $\mu(\delta) \leq \epsilon$. Since $\mu(s)$ is increasing we get that $0 \leq \mu(s) \leq \epsilon$ if $0 \leq s \leq \delta$.

To see that μ is continuous is a bit of an unnecessary nuisance; most texts don't require continuity. But to see the above $\mu(s)$ is continuous on $\mathbb{R}$, given $\epsilon > 0$ we take the δ for the uniform continuity condition. Then we check that, for all $s \in [0, \infty)$, $\mu(s + \delta) \leq \mu(s) + \epsilon$, which will imply continuity since μ is increasing.

To check this, if $|x - y| \leq s + \delta$ there is some z with $|z - x| \leq s$ and $|y - z| \leq \delta$. Thus,

$$|f(x) - f(y)| \leq |f(x) - f(z)| + |f(z) - f(y)| \leq \mu(s) + \epsilon.$$

Then

$$\mu(s+\delta) = \sup\big\{\, |f(x)-f(y)| : |x-y| \le s+\delta \big\} \le \mu(s) + \epsilon.$$

We can make $\mu(s)$ strictly increasing by noting that $\mu(s)+s$ is also a modulus of continuity for f.

(b.) The proof is essentially identical for uniform equicontinuity, just define

$$\mu(s) = \sup\big\{\, |f(x)-f(y)| : |x-y| \le s,\ f \in \mathcal{F} \big\}.$$

Pugh, Ch. 4: 19.

By density, the balls $M_\delta a$ for $a \in A$ cover M, so choosing a finite subcover we get a finite collection so that $M \subset \bigcup_{j=1}^N M_\delta a_j$.

Additional Problem 1: Consider the equality, for $|x| < R$,

$$C\Big(1-\frac{x}{R}\Big)^{-1} = \sum_{k=0}^{\infty} C\, R^{-k}\, x^k\,.$$

Differentiate m times to get

$$\frac{C\, m!}{R^m}\Big(1-\frac{x}{R}\Big)^{-m-1} = \sum_{k=m}^{\infty} \frac{k!}{(k-m)!}\, C\, R^{-k}\, x^{k-m}\,.$$

Now consider

$$f(x) = \sum_{k=0}^{\infty} a_k\, x^k\,, \qquad f^{(m)}(x) = \sum_{k=m}^{\infty} \frac{k!}{(k-m)!}\, a_k\, x^{k-m}\,.$$

By comparison,

$$\begin{aligned} |f^{(m)}(x)| &\le \sum_{k=m}^{\infty} \frac{k!}{(k-m)!}\, |a_k|\, |x|^{k-m} \\ &\le \sum_{k=m}^{\infty} \frac{k!}{(k-m)!}\, C R^{-k}\, |x|^{k-m} \\ &= \frac{C\, m!}{R^m}\Big(1-\frac{|x|}{R}\Big)^{-m-1}. \end{aligned}$$