Math 425, Winter 2018, Homework 5 Solutions

Pugh, Ch. 4: 26. One example is $M = (0, \infty)$ with the standard metric, and $f(x) = \frac{1}{2}x$.

Pugh, Ch. 4: 27.

(a.) A weak contraction does not need to be a contraction: for an example we will take a continuously differentiable function $f(x)$ on $\mathbb{R}$ so that $|f'(x)| < 1$ for all x, with $0 < f'(x) < 1$ and $\lim_{x\to-\infty} f'(x) = 1$, say

$$f(x) = \log(1 + e^x)\,, \qquad f'(x) = \frac{e^x}{1 + e^x}.$$

If it holds that $|f(x) - f(y)| \le L|x - y|$ then we have $|f'(x)| \le L$ wherever $f'(x)$ exists, so this cannot hold for the above map with $L < 1$. Note that this map does not have a fixed point on $\mathbb{R}$, since $x = \log(1 + e^x)$ would give $e^x = 1 + e^x$.

(b.) Even on a compact set we can have a weak contraction that is not a contraction by the above method: let $f(x) = \frac{1}{2}x^2$ on $[0, 1]$. Then $f'(x) = x$ has limit 1 at $x = 1$ so is not a contraction, but if $x \ne y$ we have

$$|\tfrac{1}{2}x^2 - \tfrac{1}{2}y^2| = \tfrac{1}{2}(x + y)\,|x - y|$$

and if $x \ne y$ then one of them is less than 1 so $\frac{1}{2}(x + y) < 1$.

(c.) The quickest proof of this is to find a minimum of the continuous function $d(x, f(x))$ on the compact set M. Let the minimum occur at x. If $x \ne f(x)$, then letting $y = f(x)$ we have

$$d(y, f(y)) = d(f(x), f(f(x))) < d(x, f(x)),$$

a contradiction. A fixed point must be unique: if $x \ne y$ and $f(x) = x, f(y) = y$, we have a contradiction: $d(x, y) = d(f(x), f(y)) < d(x, y)$.

A more illustrative proof depends on the fact that, if K is a compact set and f a weak-contraction, then $\operatorname{diam}(f(K)) < \operatorname{diam}(K)$ unless $f(K)$ is a single point. To see this, since K is compact so is $f(K)$, and so if $\operatorname{diam}(f(K)) > 0$ then there exist points $x, y \in f(K)$ so that $\operatorname{diam}(f(K)) = d(x, y)$. If we write $x = f(x')$ and $y = f(y')$, for $x', y' \in K$, then $d(x', y') > d(x, y)$, so $\operatorname{diam}(K) > \operatorname{diam}(f(K))$.

The quickest proof now is to consider the nested sequence of sets $M_j = f^j(M)$; that is, $M_0 = M$ and $M_{j+1} = f(M_j)$. These are nested non-empty compact sets, so $K = \bigcap_{j=1}^{\infty} M_j$ is non-empty. But $f(K) = K$. So K must be a single point, hence a fixed point.

Pugh, Ch. 4: 34.

(a.)-(b.) The point here is just to verify that if $c \ge 0$, then the function

$$y(t) = \begin{cases} 0, & t \le c, \\ (t - c)^2, & t \ge c \end{cases}$$

satisfies $y'(t) = 2\sqrt{|y(t)|}$. This is true for $t < c$ since both sides equal 0, and similarly for $t > c$ since both sides equal $(t - c)$. We thus need to verify that $y'(c) = 0$ exists and $= 0$. For this, we observe that

$$\left|\frac{y(t) - y(c)}{t - c}\right| \le |t - c| \to 0 \quad \text{as } t \to c.$$

(c.) The Picard Theorem assumes that $F(y)$ is Lipschitz. But $2\sqrt{|y|}$ is not Lipschitz at $y = 0$, since that would require $\sqrt{|y|} \leq L|y|$ for some constant L. However, taking $y = \epsilon^2$ this only holds for $\epsilon \geq L^{-1}$.

Additional problem 1. Suppose this holds for f. Given $\epsilon > 0$ use Weirstrass Approximation to find $q(x)$ so $|f(x) - q(x)| < \epsilon$ for all $x \in [a, b]$. Then

$$\epsilon^2(b-a) \geq \int_a^b |f(x) - q(x)|^2\, dx = \int_a^b f(x)^2 + q(x)^2\, dx - 2\int_a^b f(x)q(x)\, dx$$

The last term is 0 by assumption, and the integral of $q(x)^2$ is nonnegative. So we get $\int_a^b |f(x)|^2\, dx \leq \epsilon^2(b-a)$ for all ϵ. This forces $\int_a^b |f(x)|^2\, dx = 0$ (since $f(x)^2$ is non-negative), and since $f(x)^2$ is continuous and non-negative this forces $f(x) = 0$ for all x.

Additional problem 2(a). Let $g(y) = f(-\log y)$. Then since $-\log y$ maps $(0, 1]$ continuously into $[0, \infty)$, we see $g \in C((0, 1])$. It is bounded since f is bounded.

Additional problem 2(b). We want to show that $|g(y)| < \epsilon$ if $0 < y < \delta$. Since $x = -\log y$ is a decreasing function of y, this is equivalent to $|f(x)| < \epsilon$ if $x > -\log \delta$. We know $|f(x)| < \epsilon$ if $x > M$, so take $\delta = e^{-M}$ to get $|g(y)| < \epsilon$ for $0 < y < \delta$. Letting $g(0) = 0$ makes g continuous on$[0, 1]$.

Additional problem 2(c). By Weirstrass, there is a polynomial $q(x)$ so $|g(y) - q(y)| < \frac{1}{2}\epsilon$ for all $y \in [0, 1]$. In particular $|q(0)| < \frac{1}{2}\epsilon$, so letting $p(y) = q(y) - q(0)$ gives a polynomial with $|g(y) - p(y)| < \epsilon$ for all $y \in [0, 1]$. Thus

$$|f(x) - p(e^{-x})| = |g(e^{-x}) - p(e^{-x})| < \epsilon \quad \text{for all } \; x \in [0, \infty).$$

Additional problem 3(a). Given $\epsilon > 0$, for each $x \in [a, b]$, there is some N_x depending on x so $f_{N_x}(x) < \frac{1}{2}\epsilon$. By continuity of f_{N_x} there is $r_x > 0$ so $f_{N_x}(y) < \epsilon$ if $|y - x| < r_x$. The neighborhoods $|y - x| < r_x$ cover $[a, b]$, so we can find a finite cover, say $[a, b] \subset \bigcup_{j=1}^m (x_j - r_j, x_j + r_j)$. Let N_j be the N_x for x_j. Take $N = \max_j N_j$. By the monotonic decreasing property, if $n \geq N$ then

$$f_n(y) \leq f_{N_j}(y) \leq \epsilon \;\text{ if }\; y \in (x_j - r_j, x_j + r_j).$$

This is true for all j, which cover $[a, b]$, so $f_n(y) < \epsilon$ for all $y \in [a, b]$ if $n \geq N$.

Additional problem 3(b). For example the functions

$$f_n(x) = \begin{cases} 0, & x \in [0, n], \\ x - n, & x \in [n, n+1], \\ 1, & x \in [n, \infty) \end{cases}.$$