Math 425 Winter 2018

Homework 6 **Due Wednesday, February 28**

Additional problems

1. In lecture we use the following fact. Suppose $F(t) = \big(f_1(t), \dots, f_n(t)\big)$ is a continuous map from $[a, b]$ into $\mathbb{R}^m$. Then

$$\Big| \int_a^b F(s)\, ds \Big| \le \int_a^b |F(s)|\, ds.$$

Prove this, by taking N large enough so that both integrals are approximated within ϵ (so N depends on ϵ) by their Riemann sum approximation with spacing $\delta = \frac{1}{N}$.

2. Consider a map $F : \mathbb{R}^m \to \mathbb{R}^m$ satisfying $|F(x) - F(y)| \le L|x - y|$. Consider the sequence of continuous functions $\gamma_j : [0, \infty) \to \mathbb{R}^m$ determined by setting $\gamma_0(t) = p$ for all t, and the following recursion relation:

$$\gamma_{j+1}(t) = p + \int_0^t F(\gamma_j(s))\, ds.$$

(a.) Let $M = |F(p)|$. Show by induction that, for all $t \ge 0$ and $j \ge 0$,

$$|\gamma_{j+1}(t) - \gamma_j(t)| \le \frac{M L^j t^{j+1}}{(j+1)!}.$$

(b.) Use that $\gamma_n(t) = p + \sum_{j=1}^n \gamma_j(t) - \gamma_{j-1}(t)$ to show that $\gamma_n(t)$ converges uniformly on the interval $[0, T]$, for each $T < \infty$. (Note: this is weaker than uniform convergence on $[0, \infty)$.)

(c.) Show that $\gamma(t) = \lim_{n\to\infty} \gamma_n(t)$ satisfies

$$\gamma(t) = p + \int_0^t F(\gamma(s))\, ds \quad \text{for all } \ t \in [0, \infty).$$

3. Let $|x|_1 = \sum_{j=1}^m |x_j|$ and $|y|_{max} = \max_i |y_i|$. For a $n \times m$ matrix A show that

$$|T_A x|_{max} \le L|x|_1 \quad \text{where} \quad L = \max_{ij} |A_{ij}|$$

and that (for every matrix A except the 0 matrix) this is the smallest value of L for which this holds. Use this to show that

$$|T_A x|_E \le L|x|_E \quad \text{where} \quad L = \sqrt{mn} \max_{ij} |A_{ij}|$$

For each $m \ge 2$, find an example with $n = m$ of A (other than the 0 matrix) where this holds with a smaller value of L, and find another example of A where this is the smallest value of L for which this holds.

4. Let A be a $m \times m$ matrix, and $T_A : \mathbb{R}^m \to \mathbb{R}^m$ the corresponding linear map. Show that the matrix A is invertible if and only if there is some $c > 0$ so that

$$(*) \qquad |T_A x|_E \geq c\,|x|_E \quad \text{for all } \; x \in \mathbb{R}^m.$$

Show that if $(*)$ is true for a positive number c, then $|T_{A^{-1}} x|_E \leq c^{-1}|x|_E$ for the same value of c.