Math 425, Winter 2018, Homework 6 Solutions

Additional problem 1. Consider the partition $x_j = a + \frac{j}{N}(b-a)$. This has spacing $(b-a)/N$ which goes to 0 as $N \to \infty$. So we can write

$$\Big|\int_a^b F(s)\,ds\Big| = \lim_{N\to\infty} \frac{(b-a)}{N}\Big|\sum_{j=1}^N F(x_j)\Big| \le \lim_{N\to\infty} \frac{(b-a)}{N}\sum_{j=1}^N |F(x_j)| = \int_a^b |F(s)|\,ds.$$

Additional problem 2(a). Start with $\gamma_0(t) = p$, and $\gamma_1(t) = p + t\,F(p)$. Then

$$|\gamma_1(t) - \gamma_0(t)| = t|F(p)| \le Mt.$$

This is the case $j = 0$. Assume it holds for some j. Then write

$$\begin{aligned}
|\gamma_{j+2}(t) - \gamma_{j+1}(t)| &= \Big|\int_0^t F(\gamma_{j+1}(s)) - F(\gamma_j(s))\,ds\Big| \\
&\le \int_0^t |F(\gamma_{j+1}(s)) - F(\gamma_j(s))|\,ds \\
&\le \int_0^t L|\gamma_{j+1}(s) - \gamma_j(s)|\,ds \\
&\le \frac{1}{(j+1)!}\int_0^t LML^j s^{j+1}\,ds = \frac{1}{(j+2)!}ML^{j+1}t^{j+2}.
\end{aligned}$$

This is the statement for $j+1$, so the result follows by induction.

Additional problem 2(b). For $t \in [0,T]$ we can bound

$$|\gamma_j(t) - \gamma_{j-1}(t)| \le \frac{1}{j!}ML^{j-1}T^j \equiv M_j.$$

The M_j are summable:

$$\sum_{j=1}^\infty \frac{1}{j!}ML^{j-1}T^j = \frac{M}{L}\big(e^{LT} - 1\big).$$

By the Weirstrass M-test the series converges uniformly on $[0,T]$, for any chosen T.

Additional problem 2(c). Let $\gamma_j(t) \to \gamma(t)$ uniformly on $[0,T]$. Then by Lipschitz continuity of F we have $F(\gamma_j(s)) \to F(\gamma(s))$, uniformly on $[0,T]$. So since integrals of uniform limits are the limit of the integrals, we conclude that

$$\gamma(t) = \int_0^t F(\gamma(s))\,ds \qquad \text{for } \ t \in [0,T].$$

However this holds for every chosen value of T, thus it holds for all $t \in \mathbb{R}$.

Additional problem 3. For any i, we have

$$|(T_A x)_i| \le \sum_{j=1}^m |A_{ij}|\,|x_j| \le \Big(\max_j |A_{ij}|\Big)\sum_{j=1}^m |x_j| = \Big(\max_j |A_{ij}|\Big)\,\|x\|_1.$$

So

$$\max_i |(T_A x)_i| \le \Big(\max_{i,j} |A_{ij}|\Big)\,\|x\|_1.$$

Take $x = e_j$, and note $\|T_A e_j\|_{\max} = \max_i |A_{ij}| \, \|e_j\|_1$. Assuming $|T_A e_j|_{\max} \le C|x|_1$ for all $x = e_j$, then we must have $C \ge \max_{i,j} |A_{ij}|$.

We then note

$$|T_A x|_E \le \sqrt{n}|T_A x|_{\max} \le \sqrt{n}\Big(\max_{i,j} |A_{ij}|\Big)|x|_1 \le \sqrt{n}\sqrt{m}\Big(\max_{i,j} |A_{ij}|\Big)|x|_E.$$

With A the identity matrix the result holds with $L = 1$, but $\sqrt{m^2}\Big(\max_{i,j} |A_{ij}|\Big) = m$.

If $A_{ij} = 1$ for all i, j, and $x = (1, 1, \ldots, 1)$, then $T_A x = (m, m, \ldots, m)$, so $|x|_E = \sqrt{m}$, $|T_A x|_E = m\sqrt{m}$, and thus

$$|T_A x|_E = \sqrt{m^2}\Big(\max_{i,j} |A_{ij}|\Big)|x|_E.$$

This example also works when $m \ne n$.

Additional problem 4. If $|T_A x|_E \ge c|x|_E$ for some $c > 0$, this implies that $T_A x = 0$ only if $x = 0$, so A is invertible. Conversely, if A is invertible, let A^{-1} be the inverse matrix. Then there is some C so that for all $y \in \mathbb{R}^m$,

$$|T_{A^{-1}} y|_E \le C|y|_E.$$

If we plug in $y = T_A x$, we get

$$|x|_E \le C|T_A x|_E \quad \Rightarrow \quad |T_A x|_E \ge C^{-1}|x|_E.$$

so $(*)$ is true with $c = C^{-1}$.

For the second part, if $|T_A x|_E \ge c|x_E|$, then setting $x = T_{A^{-1}} y$ gives

$$|y|_E \ge c\,|T_{A^{-1}} y|_E \quad \Rightarrow \quad |T_{A^{-1}} y|_E \le c^{-1}|y|_E.$$