Math 425 — Winter 2018

Homework 7 — **Due Wednesday, March 7**

Chapter 5 [Pugh, pg. 366]: 7(c), 8, 17(a), 18.

Additional problems

1. Consider the space of $n \times n$ matrices $\mathcal{M} = \mathbb{R}^{n^2}$, that is, n^2 dimensional Euclidean space. Show that the set of invertible matrices is an open, dense subset of $\mathcal{M}$. [Use the fact that $\det(A)$ is a polynomial in the coefficients of A, and the invertible matrices are those with $\det(A) \neq 0$.]

Some help on Problem 8(c). Showing that the set of functions $f_n(t)$ is closed in $C([0,1])$ is the trickiest part. It would be easier to show they are closed in $C([0,2\pi])$ so you can do this problem on $[0,2\pi]$ instead of $[0,1]$ if you prefer.

The idea is that the set f_n satisfies $\|f_n - f_m\|_u \geq c$, for the same $c > 0$ for all $m \neq n$. That is, there is some t so $|f_m(t) - f_n(t)| \geq c$ if $m \neq n$. The easiest way to prove this is indirectly, by showing that

$$\int_0^1 |f_n(t) - f_m(t)|^2 \, dt \geq c^2,$$

for some $c > 0$ and all $m \neq n$. This is a direct calculation using double-angle formulas and product rules for cos. If you integrate over $[0,2\pi]$ instead you get equality with $c^2 = 2\pi$, which is why it's easier in that case. If you want to do it for $[0,1]$, then it is easier to find a value of c that works for $m, n \geq 100$, and write your set as the union of two closed sets.

This proves a lower bound on $\|f_n - f_m\|_u$, since

$$\int_a^b |f_n(t) - f_m(t)|^2 \, dt \leq (b-a)\|f_n - f_m\|_u^2.$$

Once you know that the points in your set $\{f_n\}$ are separated, the only sequences in this set that can converge must be eventually constant, so the limit will be in the set.