Math 425, Winter 2018, Homework 7 Solutions

Pugh, Ch. 5: 7(c). Consider the function

$$f_n(t) = \begin{cases} 1 - nx, & 0 \le x \le \frac{1}{n} \\ 0, & \frac{1}{n} \le x \le 1 \end{cases}$$

Then $\max |f(t)| = 1$, but $\int_0^1 |f(t)|\, dt = \frac{1}{2n}$. So $|f_n|_1 \to 0$ but $|f_n|_u = 1$.

Pugh, Ch. 5: 8(a). Linearity follows from additivity of the integral. For continuity:

$$\Big| \int_0^x f(t)\, dt \Big| \le \int_0^x |f(t)|\, dt \le \int_0^1 |f(t)|\, dt \le |f|_{C^0}.$$

It follows that $\max_x |T(f)(x)| \le |f|_{C^0}$. If $f(t) = 1$ then $T(f) = x$, and $|x|_{C^0} = 1 = |1|_{C^0}$, so the norm of the operator is 1.

Pugh, Ch. 5: 8(b). If $f_n = \cos(nt)$ then $T(f_n) = \frac{1}{n}\sin(nt)$.

Pugh, Ch. 5: 8(c). The set K is bounded since $|f_n|_{C^0} \le 1$ for all n. It is not compact since it is not equicontinuous: to see it is not equicontinouos, note that for any open interval (a, b) the function f_n takes on both 1 and -1 as values if n is large enough, hence there is no δ so that $|f_n(t) - f_n(x)| < \frac{1}{2}$ for every n and every $t \in (x - \delta, x + \delta)$. To see that K is closed lets work on $[0, 2\pi]$, where it holds that $|f_n - f_m|_{C^0} \ge (2\pi)^{-\frac{1}{2}}$ if $m \ne n$. This is done by a calculation

$$\int_0^{2\pi} |\cos(nt) - \cos(mt)|^2\, dt = 2\pi \quad \text{if} \;\; m \ne n.$$

It follows that necessarily $|\cos(nt) - \cos(mt)|_{C^0} \ge 1$ for some t, since the integral of $|g|^2$ is less than $2\pi\, |g|^2_{C^0}$.

Once we have that $|f_n - f_m|_{C^0} \ge c$ for some $c > 0$ (which also holds on $[0, 1]$ but it's a harder calculation), then we see that the set K is closed. For if g_k is a sequence in K, that is $g_k(t) = \cos(n_k t)$ for some n_k depending on k, then for g_k to be convergent to some g we must have n_k constant for k large, say $n_k = n$, so if g_k converges to g then $g = \cos(nt) \in K$.

Pugh, Ch. 5: 8(d). $T(K)$ is bounded, since $|T(f_n)|_{C^0} \le \frac{1}{n}$. It is also equicontinuous, since $|T(f_n)'|_{C^0} = |f_n|_{C^0} \le 1$, so $T(f_n)$ is Lipschitz with constant 1 for every n. Thus $T(K)$ has a compact closure, since the closure of a bounded, equicontinuous family of functions is bounded and equicontinuous (and closed), hence compact by Arzela-Ascoli. $T(K)$ is not compact since it is not closed, as $T(f_n) \to 0$ but $0 \notin T(K)$.

You can also verify directly that the closure of $T(K)$ is $T(K) \cup \{0\}$, which then is closed and equicontinuous and bounded, hence compact. Verifying this uses a similar argument to proving that K is closed: $|T(f_n) - T(f_m)| \ge c_n$ for some $c_n > 0$ for all $m \ne n$, so the only limit of a convergent sequence in $T(K)$ is either one of the elements $T(f_n)$ (if the sequence is eventually constant) or 0 (if there are infinitely many distinct points in the sequence).

Pugh, Ch. 5: 17(a). A simple calculation gives $(Df)_t = (-\sin t, \cos t)$, which never equals $(0, 0)$. But $f(2\pi) - f(0) = (0, 0)$.

Pugh, Ch. 5: 18(a). This is just Theorem 5.

Pugh, Ch. 5: 18(b). Let $u = (u_x, u_y)$. Then

$$f(tu) = \frac{t^4\, u_x^3\, u_y}{t^4\, u_x^4 + t^2\, u_y^2}.$$

If $u_y \neq 0$, then $|f(tu)| \leq t^2\, u_x^3/u_y$, so

$$\lim_{t\to 0} \frac{f(tu) - f(0)}{t} = 0.$$

Thus $(Df)_0(u) = 0$ if $u_y \neq 0$. If $u_y = 0$, then $f(tu) = 0$, so also $(Df)_0(u) = 0$.

To see f is not differentiable at 0, note that if it were then necessarily $(Df)_0 = 0$ by the above. Consider the curve $x = t$, $y = t^2$. By the chain rule, the function $g(t) = f(t, t^2)$ would be differentiable in t at $t = 0$, with derivative $g'(0) = 0$ there. But $f(t, t^2) = \frac{1}{2}t$, so $g'(0) = \frac{1}{2}$.

Additional problem 1. We let A denote a variable in $\mathbb{R}^{m^2}$, thought of as a real matrix. Then $\det(A)$ is a polynomial on $\mathbb{R}^{m^2}$, that is, a sum of products of powers of coordinates x_i for $1 \leq i \leq m^2$. Hence it is continuous, and the set $\det(A) \neq 0$ is an open set (the preimage of $s \neq 0$ under $A \to \det(A)$. To show the set $\det(A) \neq 0$ is dense we show that the complement, $\{A : \det(A) = 0\}$, cannot contain an open set. For this, it suffices to show that if a polynomial on $\mathbb{R}^N$ vanishes on an open set then it is identically 0 (letting $N = m^2$).

Note that if $\det(A) = 0$ on a neighborhood of some A_0, then $\det(A + A_0) = 0$ on a neighborhood of 0. It is also a polynomial (since it is the translate of a polynomial), so we now need show that if a polynomial $p(x)$ vanishes on a neighborhood of 0 then it vanishes everywhere.

There are two ways to see this. First, if we write

$$p(x) = \sum_{i_1=0}^{n} \sum_{i_2=0}^{n} \cdots \sum_{i_N=0}^{n} p_{i_1,i_2,\ldots,i_N} x_1^{i_1} \cdots x_N^{i_N}$$

Then we can find the coefficients by differentiating and evaluating at $x = 0$,

$$p_{i_1,i_2,\ldots,i_N} = \frac{1}{i_1! \cdots i_N!} \partial_{x_1}^{i_1} \cdots \partial_{x_N}^{i_N} p(x)|_{x=0}.$$

But if $p(x)$ vanishes on a neighborhood of 0 then all of its derivatives vanish at 0.

Alternatively, for each $x \in \mathbb{R}^N$ consider the one dimensional polynomial $p(tx)$ for $t \in \mathbb{R}$. This vanishes on a neighborhood of $t = 0$, and since a non-zero polynomial can have at most a finite number of real roots, it must be 0 for all t. Setting $t = 1$ gives $p(x) = 0$.