Math 425 Midterm Solutions, Winter 2018

1. Find the radius of convergence of the series $\sum_{k=0}^{\infty}\big(2+(-1)^k\big)^k x^{2k}$.

Solution. The coefficient a_k of x^k is 0 if k is odd, and $a_k = 3^{\frac{k}{2}}$ for k divisible by 4, and $a_k = 2$ for k even but not divisible by 4. So

$$R = \limsup_{k\to\infty} |a_k|^{1/k} = 3^{\frac{1}{2}}.$$

2. Suppose that $g : [0,1] \to \mathbb{R}$ is a continuous function such that $g(1) = 0$. Show that:

$$x^n g(x) \to 0 \;\text{ uniformly on }\; [0,1]$$

Solution. Given $\epsilon > 0$, find $r < 1$ so that $|g(x)| < \epsilon$ if $x \in [r,1]$. Then $|x^n g(x)| < \epsilon$ for $x \in [r,1]$ since $|x^n| < 1$ there.
For $x \in [0,r]$, we have $|x^n g(x)| < M\,r^n$, where $M = \max|g(x)|$. Since $r < 1$, there is N so that $M\,r^n < \epsilon$ if $n \geq N$. So if $n \geq N$ we have $|x^n g(x)| < \epsilon$ for all $x \in [0,1]$.

3. Let $\mathcal{F}$ be the set of continuous functions $f : [0,1] \to \mathbb{R}$ that satisfy

$$f(0) = 0, \quad |f(x) - f(y)| \leq |x-y|^{\frac{1}{2}} \;\text{ for all } x, y \in [0,1].$$

Show that $\mathcal{F}$ is a compact subset of $C^0([0,1])$ with the uniform norm.
Solution. By Arzela-Ascoli we need show that $\mathcal{F}$ is a bounded, equicontinuous, and closed subset of $C^0([0,1])$.
Bounded. This means there is some M so that $\|f\|_u \leq M$ for every $f \in \mathcal{F}$. This is true with $M = 1$ since $f(0) = 0$ and $|f(x) - f(0)| \leq |x-0|^{\frac{1}{2}} \leq 1$.
Equicontinuous. Given $\epsilon > 0$ let $\delta = \epsilon^2$. Then $|x-y| < \delta$ gives, for every $f \in \mathcal{F}$, $|f(x) - f(y)| \leq |x-y|^{\frac{1}{2}} < \epsilon$.
Closed. We need show that if $f_n \in \mathcal{F}$ and $\|f - f_n\|_u \to 0$ then $f \in \mathcal{F}$. It is easy to see that $f(0) = 0$, since $f_n(0) = 0$ for all n. To check the other condition, consider any $x, y \in [0,1]$ and write

$$|f(x) - f(y)| = \lim_{n\to\infty} |f_n(x) - f_n(y)| \leq |x-y|^{\frac{1}{2}},$$

where we use the condition $f_n \in \mathcal{F}$ for all n.

4. Assume that $f(x) : [0, \infty) \to [0, \infty)$ is a non-negative, decreasing continuous function, that is $f(x) \geq f(y) \geq 0$ if $0 \leq x \leq y$, and assume that

$$\int_0^\infty f(x)\, dx \; = \lim_{n\to\infty} \int_0^n f(x)\, dx \quad \text{converges.}$$

Show that the series $g(x) = \displaystyle\sum_{n=0}^{\infty} f(x+n)$ converges uniformly for $x \in [0,1]$, and that

$$\int_0^1 g(x)\, dx \; = \; \int_0^\infty f(x)\, dx\,.$$

Solution. Let $M_k = f(k)$. Then $|f(x+k)| = f(x+k) \leq f(k)$ for $x \in [0,1]$, so the Weirstrass M-test will show that $\sum_{k=0}^{\infty} f(x+k)$ converges uniformly on $[0,1]$ if we show that $\sum_{k=0}^{\infty} f(k)$ converges. We can use the integral test and note that

$$0 \leq f(k) \leq \int_{k-1}^{k} f(x)\, dx \quad \Rightarrow \quad \sum_{k=1}^{n} f(k) \leq \int_0^n f(x)\, dx.$$

Since the right hand side is increasing in n and converges, so does $\sum_{k=1}^{\infty} f(k)$, and consequently so does $\sum_{k=0}^{\infty} f(k)$.

Finally, since we have uniform convergence of the sum,

$$\int_0^1 g(x)\, dx = \int_0^1 \Big(\sum_{k=0}^{\infty} f(x+k)\Big)\, dx = \sum_{k=0}^{\infty} \int_0^1 f(x+k)\, dx = \sum_{k=0}^{\infty} \int_k^{k+1} f(x)\, dx$$

$$= \lim_{n\to\infty} \int_0^n f(x)\, dx = \int_0^\infty f(x)\, dx.$$