Lecture 10: The Cauchy-Riemann equations

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Cauchy-Riemann equations.

We will write $w = x + iy$, and express

$$f(x + iy) \ = \ u(x, y) + iv(x, y)$$

where $u(x, y)$ and $v(x, y)$ are real-valued functions on $\mathbb{R}^2$.

Consider $z = w + h$, where h is a real number. Then

$$\frac{f(z) - f(w)}{z - w} = \frac{u(x + h, y) - u(x, y)}{h} + i\,\frac{v(x + h, y) - v(x, y)}{h}$$

If f is differentiable at w, taking the limit as $h \to 0$ gives

$$f'(x + iy) \ = \ \partial_x u(x, y) \ + \ i\,\partial_x v(x, y)\,.$$

Consider $z = w + ih$, where h is a real number. Then

$$\frac{f(z) - f(w)}{z - w} = \frac{u(x, y+h) - u(x, y)}{ih} + i\,\frac{v(x, y+h) - v(x, y)}{ih}$$

If $f'(x+iy)$ exists, then taking the limit as $h \to 0$ gives

$$f'(x+iy) = -i\partial_y u(x,y) + \partial_y v(x,y)\,.$$

Thus: $\partial_x u(x,y) + i\partial_x v(x,y) = -i\partial_y u(x,y) + \partial_y v(x,y)$, so

Cauchy-Riemann equations: if $f = u + iv$ is analytic, then

$$\partial_x u(x,y) = \partial_y v(x,y)\,, \qquad \partial_y u(x,y) = -\partial_x v(x,y)\,.$$

C-R equations imply analyticity

Theorem: suppose $f(x+iy) = u(x,y) + iv(x,y)$

If u and v are differentiable on E, then f is analytic on E if

$$\partial_x u(x,y) = \partial_y v(x,y)\,, \qquad \partial_y u(x,y) = -\partial_x v(x,y)\,.$$

Example:

$$e^{x+iy} = e^x\left(\cos y + i\sin y\right)$$

$$u(x,y) = e^x\cos y\,, \qquad v(x,y) = e^x\sin y\,.$$

$$\partial_x u(x,y) = e^x\cos y = \partial_y v(x,y)$$

$$\partial_y u(x,y) = -e^x\sin y = \partial_x v(x,y)$$

The derivative of $\arg(z)$

Lemma

For any branch of $\arg(x, y) = \arg(x + iy)$, on $\mathbb{C} \setminus \{\text{cut-line}\}$

$$\partial_x \arg(x, y) = \frac{-y}{x^2 + y^2}, \qquad \partial_y \arg(x, y) = \frac{x}{x^2 + y^2}.$$

Proof. It suffices to prove it near each point for some branch, since different branches differ by a constant $2\pi k$.

- For $x > 0$ choose $\arg(x, y) = \arctan(y/x)$.
- For $x < 0$ choose $\arg(x, y) = \arctan(y/x) + \pi$.
- For $y > 0$ choose $\arg(x, y) = \operatorname{arccot}(x/y)$.
- For $y < 0$ choose $\arg(x, y) = \operatorname{arccot}(x/y) + \pi$.

Derivative of $\log z$

Principal branch: $\log(z) = \log|z| + i \arg_{(-\pi,\pi]}(z)$

$$u(x,y) = \tfrac{1}{2}\log(x^2+y^2)\,, \qquad v(x,y) = \arg_{(-\pi,\pi]}(x,y)\,.$$

C-R holds, partial derivatives are continuous on $\mathbb{C} \setminus (-\infty, 0]$, so

$$(\log z)' = \partial_x u + i\partial_x v = \frac{x}{x^2+y^2} + i\,\frac{-y}{x^2+y^2} = \frac{1}{x+iy}$$

$$(\log z)' = \frac{1}{z}$$

This rule holds for every branch of $\log z$, off its cut-line.

Derivative of $z^{\frac{1}{2}}$

Principal branch: $z^{\frac{1}{2}} = e^{\frac{1}{2}\log z}$ (principal branch of log)

By chain rule:

$$\left(z^{\frac{1}{2}}\right)' = e^{\frac{1}{2}\log z} \cdot \frac{1}{2}\,(\log z)' = \frac{1}{2}\, z^{\frac{1}{2}}\, z^{-1}$$

Writing $z^{-1} = e^{-\log z}$, we see $z^{\frac{1}{2}}\, z^{-1} = e^{-\frac{1}{2}\log z}$,

$$\left(z^{\frac{1}{2}}\right)' = \frac{1}{2}\, z^{-\frac{1}{2}}$$

where $z^{-\frac{1}{2}}$ is also the principal branch.