Lecture 13: Cauchy's Theorem

Hart Smith

Department of Mathematics
University of Washington, Seattle

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More on contour integrals

Useful notation:

- If $z_0, z_1 \in \mathbb{C}$, $[z_0, z_1]$ = straight line path from z_0 to z_1

$$[z_0, z_1] = \{(1-t)z_0 + t z_1 \ : \ t \in [0,1]\}$$

We will write $$\int_{[z_0,z_1]} f(z)\,dz = \int_{z_0}^{z_1} f(z)\,dz$$

- For γ a path, $-\gamma$ denotes the same path in the other direction

$$-\gamma(t) = \{\gamma(-t) \ : \ t \in [-b,-a]\}$$

Then $\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz$, and $-[z_0, z_1] = [z_1, z_0]$.

Contour integrals over the boundary of a domain

We integrate over boundaries in counter-clockwise direction:

- If $\Delta = $ triangle connecting $\{z_0, z_1, z_2\}$ counter-clockwise

$$\int_{\partial\Delta} f(z)\,dz = \int_{z_0}^{z_1} f(z)\,dz + \int_{z_1}^{z_2} f(z)\,dz + \int_{z_2}^{z_0} f(z)\,dz$$

Equivalently: $\partial\Delta$ is the contour $[z_0, z_1] \cup [z_1, z_2] \cup [z_2, z_0]$

- If $D_r(w) = $ disc of radius r centered at w,

$$\int_{\partial D_r(w)} f(z)\,dz = \int_0^{2\pi} f(w + re^{it})\, ire^{it}\,dt$$

Equivalently: $\partial D_r(w)$ is the contour $\{w + re^{it} : t \in [0, 2\pi]\}$

Cauchy's Theorem

Cauchy's Theorem for a triangle Δ

If $f(z)$ is a smooth analytic function on an open set $E \supset \Delta$, then

$$\int_{\partial\Delta} f(z)\,dz \;=\; 0$$

Relate to real line integrals: if $f(x+iy) \;=\; u(x,y)+i\,v(x,y)$,

$$\begin{aligned}\int_\gamma f(z)\,dz \;&=\; \int_\gamma \Big(u(x,y)+i\,v(x,y)\Big)\,(dx+i\,dy)\\ &=\; \int_\gamma \Big(u\,dx \;-\; v\,dy\Big) \;+\; i\int_\gamma \Big(v\,dx+u\,dy\Big)\end{aligned}$$

Green's Theorem for a triangle Δ

If u, v continuously differentiable on an open set $E \supset \Delta$, then

$$\int_{\partial\Delta} v\,dx + u\,dy \;=\; \int_{\Delta}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

Replacing (u, v) with $(-v, u)$ gives:

$$\int_{\partial\Delta} u\,dx - v\,dy \;=\; -\int_{\Delta}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dx\,dy$$

Cauchy's Theorem follows from **Cauchy-Riemann equations**:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad\qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Arclength of a path

For a smooth curve in $\mathbb{R}^2$ given by $\{\,(x(t), y(t)) : t \in [a,b]\,\}$

$$\textbf{Distance traveled} \;=\; \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\; dt$$

If $\gamma(t) = x(t) + i\,y(t)$, then $|\gamma'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$,

$$\textbf{Distance traveled by } \gamma : \quad \ell(\gamma) \;=\; \int_a^b |\gamma'(t)|\, dt$$

Semicircle: $\gamma(t) = e^{it}$, $t \in [0,\pi]$, $\ell(\gamma) = \pi$.

Recall the inequality: $\left| \int_a^b g(t)\,dt \right| \le \int_a^b |g(t)|\,dt$, we get

$$\left| \int_\gamma f(z)\,dz \right| \le \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt$$

Corollary

If $|f(z)| \le M$ for all z in the image of γ, then

$$\left| \int_\gamma f(z)\,dz \right| \le M \cdot \ell(\gamma)$$

Typical application:

$$\left| \int_{\partial D_r(z_0)} f(z)\,dz \right| \le 2\pi r \cdot \max |f(z)| .$$