Lecture 14: Cauchy's theorem, anti-derivatives on convex sets

Hart Smith

Department of Mathematics
University of Washington, Seattle

Math 427, Autumn 2019

Last lecture

Cauchy's Theorem for a triangle Δ

If $f(z)$ is a smooth analytic function on an open set $E \supset \Delta$, then

$$\int_{\partial\Delta} f(z)\,dz \;=\; 0$$

Definition: $E \subset \mathbb{C}$ is convex if $[z_0, z_1] \subset E$ whenever $z_0, z_1 \in E$.

Corollary

Let E be an open, convex set in $\mathbb{C}$, and z_0, z_1, z_2 be points in E. If f is a smooth analytic function on E, then

$$\int_{z_0}^{z_1} f(z)\,dz + \int_{z_1}^{z_2} f(z)\,dz = \int_{z_0}^{z_2} f(z)\,dz$$

Theorem

Suppose E is an open, convex set, and $f(z)$ is analytic on E.

Fix $z_0 \in E,\ c_0 \in \mathbb{C},$ and define: $\quad F(z) = c_0 + \int_{z_0}^{z} f(w)\,dw\,.$

Then $F(z)$ is analytic, and $F'(z) = f(z)\,.$

Proof. By the Cauchy integral theorem for triangles

$$\begin{aligned} F(z+\lambda) - F(z) &= \int_{z_0}^{z+\lambda} f(w)\,dw - \int_{z_0}^{z} f(w)\,dw \\ &= \int_{z}^{z+\lambda} f(w)\,dw \end{aligned}$$

So we need to show that

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \int_{z}^{z+\lambda} f(w)\,dw = f(z)\,.$$

$[z, z+\lambda]$ is given by the curve γ, where

$$\gamma(t) \;=\; \{z + t\lambda \;:\; t \in [0,1]\}\,, \qquad \gamma'(t) = \lambda\,.$$

So:

$$\begin{aligned}\left|\frac{1}{\lambda}\int_z^{z+\lambda} f(w)\,dw \;-\; f(z)\right| \;&=\; \left|\frac{1}{\lambda}\int_0^1 f(z+t\lambda)\,\lambda\,dt \;-f(z)\right| \\ &=\; \left|\int_0^1 \Big(f(z+t\lambda) - f(z)\Big)\,dt\right|\end{aligned}$$

By continuity of f, if $|\lambda| < \delta$ then $\big|\,f(z+t\lambda) - f(z)\,\big| \;<\; \epsilon$, so

$$\left|\int_0^1 \Big(f(z+t\lambda) - f(z)\Big)\,dt\right| \le \int_0^1 \big|\,f(z+t\lambda) - f(z)\,\big|\,dt \;<\; \epsilon$$

Cauchy's theorem for convex sets

Suppose that $f(z)$ is analytic on an open convex set E.

If γ is a closed path in E, then $\displaystyle\int_\gamma f(z)\,dz \;=\; 0\,.$

Proof. Write $f(z) = F'(z)$; use FTC and $\gamma(b) = \gamma(a)$:

$$\int_\gamma f(z)\,dz = F\big(\gamma(b)\big) - F\big(\gamma(a)\big) = 0$$

Corollary: assume $E \subset \mathbb{C}$ is convex, f analytic on E

If $\gamma : [a, b] \to E$ and $\mu : [a', b'] \to E$ are paths with $\gamma(a) = \mu(a')$, and $\gamma(b) = \mu(b')$, then $\displaystyle\int_\gamma f(w)\,dw = \int_\mu f(w)\,dw$.

Thus, if $f(z)$ is analytic on E, then we can calculate $F(z)$ by

$$F(z) \;=\; F(z_0) \;+\; \int_\gamma f(w)\,dw$$

for any convenient choice of γ connecting z_0 to z.

Example. Anti-derivative of z^n on $\mathbb{C}$.

Set $z_0 = 0$. We take $F(0) = 0$, and define

$$F(z) = \int_0^z w^n \, dw$$

following the path $\gamma(t) = tz\,, \quad t \in [0,1]\,, \quad$ so $\gamma'(t) = z\,.$

$$\int_0^z w^n \, dw = \int_0^1 (tz)^n \, z \, dt = z^{n+1} \int_0^1 t^n \, dt = \tfrac{1}{n+1} \, z^{n+1}$$

Example. Anti-derivative of z^{-1} on $E = \{z : \mathrm{Re}(z) > 0\}$.

Set $z_0 = 1$. We take $F(1) = 0$, and define

$$F(z) = \int_\gamma w^{-1}\, dw$$

where γ follows the circle $\{|w| = 1\}$ from 1 to $e^{i\theta}$, $\theta = \arg(z)$, then along the radial line from $e^{i\theta}$ to $z = |z|e^{i\theta}$.

- From 1 to $e^{i\theta}$: $\displaystyle\int_0^\theta \left(e^{it}\right)^{-1} i\, e^{it}\, dt = \int_0^\theta i\, dt = i\theta\,.$
- $e^{i\theta}$ to $|z|e^{i\theta}$: $\displaystyle\int_1^{|z|} \left(te^{i\theta}\right)^{-1} e^{i\theta}\, dt = \int_1^{|z|} t^{-1}\, dt = \log|z|\,.$

$$\int_1^z w^{-1}\, dw = \log|z| + i\arg(z)\,.$$