Lecture 15: Cauchy's Integral Formula

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Review: Cauchy's Theorem for a triangle Δ

If $f(z)$ is analytic on a convex open set E, then for any $\Delta \subset E$,

$$\int_{\partial\Delta} f(z)\,dz \;=\; 0$$

Anti-derivatives and Cauchy's Theorem on convex sets

Let $E \subset \mathbb{C}$ be open, convex, and $f(z)$ continuous on E. If

$$\int_{\partial\Delta} f(z)\,dz \;=\; 0 \quad \text{for all } \;\Delta \subset E$$

then $F(z) \;=\; \displaystyle\int_{z_0}^{z} f(w)\,dw$ is analytic on E, and $F'(z) = f(z)$.

Therefore, $\displaystyle\int_{\gamma} f(z)\,dz \;=\; 0$ for all closed paths $\gamma \subset E$.

Generalization: assume E open, convex, and $z_0 \in E$.

Suppose $g(z)$ is continuous on E, and analytic on $E \setminus \{z_0\}$.

Then $\displaystyle\int_{\partial\Delta} g(z)\,dz = 0$ for all $\Delta \subset E$, so $g(z) = G'(z)$,

$$\text{and} \quad \int_\gamma g(z)\,dz = 0 \quad \text{for all closed paths } \gamma \subset E\,.$$

Example of interest. Assume $f(z)$ analytic on E, and $z_0 \in \mathbb{C}$.

$$g(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0}\,, & z \in E \setminus \{z_0\} \\[2ex] f'(z_0)\,, & z = z_0 \end{cases}$$

Then $\displaystyle\int_\gamma g(w)\,dw = 0$ for every closed path γ in E.

If E is convex open, γ a closed path in E that does not touch z_0,

$$\int_\gamma \frac{f(w) - f(z_0)}{w - z_0}\, dw = 0$$

Cauchy integral formula

$$\int_\gamma \frac{f(w)}{w - z_0}\, dw = f(z_0) \cdot \left(\int_\gamma \frac{1}{w - z_0}\, dw \right)$$

Index Theorem

If γ is a closed path in $\mathbb{C}$ that does not touch z_0, then

$$\int_\gamma \frac{1}{w - z_0}\, dw = 2\pi i\, k$$

for some integer k. We call k the index of z_0 with respect to γ,

$$\operatorname{ind}_\gamma(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{w - z_0}\, dw$$

Proof of the Index Theorem

$$\int_\gamma \frac{1}{w - z_0}\, dw = \int_a^b \frac{\gamma'(t)}{\gamma(t) - z_0}\, dt = \int_a^b \frac{\mu'(t)}{\mu(t)}\, dt$$

where $\mu(t) = \gamma(t) - z_0$, so $\mu(t)$ is a closed path in $\mathbb{C} \setminus \{0\}$.

Intuition: if μ a path in $\mathbb{C} \setminus (-\infty, 0]$ (maybe not closed)

$$\int_a^b \frac{\mu'(t)}{\mu(t)}\, dt = \int_a^b \frac{d}{dt} \log(\mu(t))\, dt = \log(\mu(b)) - \log(\mu(a)).$$

Imaginary part is $\arg(\mu(b)) - \arg(\mu(a))$.

- If $\mu(t)$ wraps around 0, i.e. $\gamma(t)$ wraps around z_0, integral requires different branches; differences in $\arg(\mu(t))$ add up.

To handle paths μ that wrap around $\{0\}$:

- Find points $a = a_0 < a_1 < \cdots < a_n = b$ so that image of $[a_j, a_{j+1}]$ by μ lies in some half-space.
- We can then write $\displaystyle\int_{a_j}^{a_{j+1}} \frac{\mu'(t)}{\mu(t)}\, dt = \log(\mu(a_{j+1})) - \log(\mu(a_j))$ for some branch of $\log(z) = \log|z| + i \arg(z)$.
- Add up from $j = 1$ to n, the imaginary part gives

$$\sum_{j=1}^{n} \arg\big(\mu(a_{j+1})\big) - \arg\big(\mu(a_j)\big) = \text{total change in } \arg\big(\mu(t)\big)$$

- The real part gives

$$\sum_{j=1}^{n} \log|\mu(a_{j+1})| - \log|\mu(a_j)| = \log|\mu(b)| - \log|\mu(a)| = 0$$