Lecture 16: The Index Function

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Math 427, Autumn 2019

From last lecture...

Theorem

If γ is a closed path in $\mathbb{C}$, and $z \notin \{\gamma\}$, then

$$\int_\gamma \frac{1}{w-z}\,dw = 2\pi i\,k$$

for some integer k. We call k the index of z with respect to γ,

$$\mathrm{ind}_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{1}{w-z}\,dw$$

- $\mathrm{ind}_\gamma(z)$ is an integer valued function defined on $\mathbb{C}\backslash\{\gamma\}$.

Alternate Proof of Theorem

Write $\gamma(t) = z + \mu(t)$ where $\mu(t) = r(t)e^{i\theta(t)}$

- $r(t) > 0$, $\theta(t)$ smooth, real valued functions on $[a, b]$.
- $r(b) = r(a)$, $e^{i\theta(b)} = e^{i\theta(a)}$, so $\theta(b) - \theta(a) = 2\pi k$.

Then: $$\frac{\mu'(t)}{\mu(t)} = \frac{r'(t)e^{i\theta(t)} + r(t)e^{i\theta(t)}\, i\theta'(t)}{r(t)e^{i\theta(t)}} = \frac{r'(t)}{r(t)} + i\theta'(t)$$

$$\int_a^b \frac{\mu'(t)}{\mu(t)}\, dt = \log(r(t))\Big|_a^b + i\theta(t)\Big|_a^b = i\,2\pi k$$

$$\mathrm{ind}_\gamma(z) = \frac{\theta(b) - \theta(a)}{2\pi}$$

Consequence

If E is convex open, γ a closed path in E such that $z \notin \{\gamma\}$, and $f(z)$ an analytic function on E,

$$\int_\gamma \frac{f(w) - f(z_0)}{w - z_0}\, dw = 0$$

$$\int_\gamma \frac{f(w)}{w - z_0}\, dw = f(z_0) \cdot \left(\int_\gamma \frac{1}{w - z_0}\, dw \right)$$

Cauchy integral formula: $f(z)$ analytic on convex open set E

Let γ be a closed path in E that does not touch z. Then

$$\int_\gamma \frac{f(w)}{w - z}\, dw = 2\pi i\, f(z) \cdot \mathrm{ind}_\gamma(z).$$

Example

Important fact

Let γ trace the unit circle: $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$.

Then: $$\text{ind}_\gamma(z) = \begin{cases} 1\,, & |z| < 1\,, \\ 0\,, & |z| > 1\,. \end{cases}$$

Cauchy integral formula for the circle

Assume $f(z)$ analytic on an open set E containing $\{z : |z| \leq 1\}$.

Then, for $|z| < 1$: $$f(z) = \frac{1}{2\pi i} \int_{|w|=1} \frac{f(w)}{w - z}\, dw$$

Theorem: for a closed path γ

The function $\operatorname{ind}_\gamma(z)$ is continuous on $\mathbb{C} \setminus \{\gamma\}$.

Proof. Suppose z_0 is distance r from $\{\gamma\}$, and $|z - z_0| < r/2$.

$$w \in \{\gamma\} : \left| \frac{1}{w-z} - \frac{1}{w-z_0} \right| = \left| \frac{z - z_0}{(w-z_0)(w-z)} \right| \leq \frac{2|z-z_0|}{r^2}$$

So :
$$\left|\operatorname{ind}_\gamma(z) - \operatorname{ind}_\gamma(z_0)\right| = \left| \frac{1}{2\pi i} \int_\gamma \frac{1}{w-z} - \frac{1}{w-z_0}\, dw \right|$$
$$\leq \frac{1}{2\pi}\, \ell(\gamma)\, \frac{2|z-z_0|}{r^2} = \frac{\ell(\gamma)}{\pi r^2} |z - z_0|$$

This implies: $\lim_{z \to z_0} \left|\operatorname{ind}_\gamma(z) - \operatorname{ind}_\gamma(z_0)\right| = 0.$