Lecture 17: Properties of the Index Function

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From Lecture 16

Theorem

If γ is a closed path in $\mathbb{C}$ that does not touch z, then

$$\operatorname{ind}_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{1}{w - z}\, dw$$

is an integer k, called the index of z with respect to γ.

Theorem: for a closed path γ

The function $\operatorname{ind}_\gamma(z)$ is continuous on $\mathbb{C} \setminus \{\gamma\}$.

Theorem: for a closed path γ

Suppose z, z_0 are two points in $\mathbb{C} \setminus \{\gamma\}$, such that there is a continuous path from z_0 to z that does not cross γ. Then

$$\mathrm{ind}_\gamma(z) = \mathrm{ind}_\gamma(z_0)$$

Proof. Let $\mu(t)$ be a continuous path, $\mu(a) = z_0$, $\mu(b) = z$.

- By composition rule, $f(t) = \mathrm{ind}_\gamma\big(\mu(t)\big)$ is continuous map.
- $f(t) : [a, b] \to \mathbb{R}$ takes on only integer values.
- $f(t)$ must be constant on $[a, b]$, since it cannot jump:

  Intermediate value theorem: If $f(b) \neq f(a)$, then $f(t)$ takes on all real values in between $f(b)$ and $f(a)$, a contradiction.

Definition: Suppose $E \subset \mathbb{C}$ is an open set, and $z, z_0 \in E$.

We say z and z_0 are in the same *component* of E if there is a continuous path from z_0 to z that is contained in E.

We write $z \sim z_0$ when they are in the same component. Then:

- If $z \sim z_0$ and $z' \sim z_0$ then $z \sim z'$.
- For all $z \in E$, we have $z \sim z$.
- If $z \sim z'$ then $z' \sim z$.

The components of E are open since E is open:

If $D_r(z) \subset E$, then $z' \sim z$ for each point $z' \in D_r(z)$.

Theorem

The function $\mathrm{ind}_\gamma(z)$ is constant on each component of $\mathbb{C} \setminus \{\gamma\}$.

Examples

- If $\gamma(t) = e^{it}, \quad t \in [0, 2\pi]$, the components are

$$\mathbb{C} \setminus \{\gamma\} = \{z : |z| < 1\} \cup \{z : |z| > 1\}$$

$$\operatorname{ind}_\gamma(z) = \begin{cases} 1\,, & |z| < 1\,, \\ 0\,, & |z| > 1\,. \end{cases}$$

- If $\gamma(t) = \begin{cases} e^{it}, & t \in [0, 2\pi], \\ 2e^{it} - 1, & t \in [2\pi, 4\pi], \end{cases}$ the components are

$$\{z : |z| < 1\} \cup \{z : |z| > 1, |z+1| < 2\} \cup \{z : |z+1| > 2\}$$

$$\operatorname{ind}_\gamma(z) = \begin{cases} 2\,, & |z| < 1\,, \\ 1\,, & |z| > 1 \text{ and } |z+1| < 2\,, \\ 0\,, & |z+1| > 2\,. \end{cases}$$

The unbounded component of $\mathbb{C}\backslash\{\gamma\}$

- There is a unique *unbounded component* of $\mathbb{C}\backslash\{\gamma\}$, which consists of all points in $\mathbb{C}\backslash\{\gamma\}$ that can be "connected to ∞" without crossing γ.

- If γ is a closed path with image contained in $\{z : |z| \leq R\}$, then the unbounded component contains $\{z : |z| > R\}$.

- $\mathrm{ind}_\gamma(z) = 0$ for all z in the unbounded component of $\mathbb{C}\backslash\{\gamma\}$.