Lecture 18: Uniform convergence

Hart Smith

Department of Mathematics
University of Washington, Seattle

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Two notions of functions converging on a set $S \subset \mathbb{C}$

Definition

A sequence of functions F_n *converges pointwise* to F on S, if for each $\epsilon > 0$ and each $w \in S$, there is N such that

$$\left|F_n(w) - F(w)\right| \le \epsilon \text{ when } n \ge N.$$

Definition

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$$\left|F_n(w) - F(w)\right| \le \epsilon \text{ for all } w \in S, \text{ when } n \ge N.$$

Key distinction:

- In pointwise convergence, N can depend on w (and ϵ).
- In uniform convergence, N depends only on ϵ, not on w.

Important example: Consider $F_n(z) = z^n$. Then:

- z^n converges pointwise to 0 on the set $\{z : |z| < 1\}$
- z^n does not converge uniformly to 0 on $\{z : |z| < 1\}$
- If $r < 1$, z^n converges uniformly to 0 on $\{z : |z| \le r\}$

  Proof. Given ϵ, there is N so that $r^N < \epsilon$, since $r < 1$

$$\text{For } |z| \le r : \quad |z^n - 0| = |z|^n \le r^n \le \epsilon \text{ if } n \ge N$$

Related example: Consider $F_n(z) = \left(\frac{z}{R}\right)^n = \frac{z^n}{R^n}$ Then:

- $\frac{z^n}{R^n}$ converges pointwise to 0 on the set $\{z : |z| < R\}$
- If $r < R$, $\frac{z^n}{R^n}$ converges uniformly to 0 on $\{z : |z| \le r\}$

Theorem

Suppose that F_n converges uniformly to F on a set E, and γ is a path contained in E. Then

$$\lim_{n\to\infty} \int_\gamma F_n(w)\,dw = \int_\gamma F(w)\,dw\,.$$

Proof.

$$\left|\int_\gamma F_n(w)\,dw - \int_\gamma F(w)\,dw\right| = \left|\int_\gamma F_n(w) - F(w)\,dw\right| \le \ell(\gamma)\cdot M$$

where $M = \max_{w\in\{\gamma\}} |F_n(w) - F(w)|\,.$

Uniform convergence: given ϵ there is N so $M < \dfrac{\epsilon}{\ell(\gamma)}$ if $n \geq N$,

$$\left|\int_\gamma F_n(w)\,dw - \int_\gamma F(w)\,dw\right| \le \epsilon \text{ for } n \geq N.$$

Theorem

Suppose that F_n converges uniformly to F on a set E, and F_n is continuous on E for every n. Then F is continuous.

Proof. This is a classic $\frac{\epsilon}{3}$ proof. Given ϵ, and $z \in E$:

- Choose n so that $|F_n(w) - F(w)| \leq \frac{\epsilon}{3}$ for all $w \in E$
- Now that n is fixed, choose δ so $|F_n(w) - F_n(z)| \leq \frac{\epsilon}{3}$

$$|F(w) - F(z)| \leq |F(w) - F_n(w)| + |F_n(w) - F_n(z)| + |F_n(z) - F(z)|$$

$$\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

Corollary

Suppose that F_n converges uniformly to F on an open convex set E, and assume that F_n is analytic on E for every n.

Then F is continuous, and

$$\int_{\partial\Delta} F(z)\,dz = 0$$

for all triangles $\Delta \subset E$.