

Lecture 19: Convergence of power series

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Radius of convergence

Basic fact about power series

For each power series $\sum_{k=0}^{\infty} a_k z^k$, with coefficients $a_k \in \mathbb{C}$,

there is a real number R with $0 \leq R \leq \infty$, such that $\sum_{k=0}^{\infty} a_k z^k$

converges if $|z| < R$, and does not converge if $|z| > R$, $z \in \mathbb{C}$.

This is the same number R such that:

- $\sum_{k=0}^{\infty} |a_k| r^k$ converges if $r < R$,
- $\sum_{k=0}^{\infty} |a_k| r^k$ does not converge if $r > R$.

In some cases you can apply the ratio test:

If $\lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|}$ exists, then $R = \left(\lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|} \right)^{-1}$

Example: $\sum_{k=0}^{\infty} (-1)^k \frac{k z^k}{3^k}$ converges $|z| < 3$, diverges $|z| \geq 3$.

$\sum_{k=0}^{\infty} (-1)^k \frac{z^k}{k!}$ converges for all z .

If only have even (or odd) terms, modify the ratio test:

If $\lim_{k \rightarrow \infty} \frac{|a_{k+2}|}{|a_k|}$ exists, then $R = \left(\lim_{k \rightarrow \infty} \frac{|a_{k+2}|}{|a_k|} \right)^{-\frac{1}{2}}$

Example: $\sum_{k=0}^{\infty} \frac{(2k+1) z^{2k}}{2^k}$ converges $|z^2| < 2$, i.e. $|z| < \sqrt{2}$

Theorem: Root test

The series $\sum_{k=0}^{\infty} a_k z^k$ converges if $\limsup_{k \rightarrow \infty} |a_k|^{\frac{1}{k}} |z| < 1$.

This holds if and only if $|z| < R$, where $R = \left(\limsup_{k \rightarrow \infty} |a_k|^{\frac{1}{k}} \right)^{-1}$.

Useful facts: $\lim_{k \rightarrow \infty} \left(\frac{1}{k!} \right)^{\frac{1}{k}} = 0$, $\lim_{k \rightarrow \infty} (k^n)^{\frac{1}{k}} = 1$ for any n .

Application: The power series

$$\sum_{k=0}^{\infty} a_k z^k \quad \text{and} \quad \sum_{k=1}^{\infty} k a_k z^{k-1}$$

have the same radius of convergence.

$$\sum_{k=0}^{\infty} \frac{(2k)! z^k}{2^{2k}(k!)^2} = 1 + \frac{1}{2}z + \frac{1}{2}\frac{3}{2}\frac{1}{2!}z^2 + \frac{1}{2}\frac{3}{2}\frac{5}{2}\frac{1}{3!}z^3 + \dots$$

Ratio test: $\lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} = \lim_{k \rightarrow \infty} \frac{(2k+2)(2k+1)}{2^2(k+1)^2} = 1; R = 1.$

$$\sum_{k \text{ odd}} (-1)^{\frac{k-1}{2}} \frac{z^k}{k} = z - \frac{1}{3}z^3 + \frac{1}{5}z^5 - \frac{1}{7}z^7 + \dots$$

Root test: $\limsup_{k \rightarrow \infty} |a_k|^{\frac{1}{k}} = \limsup_{k \text{ odd}} (k^{-1})^{\frac{1}{k}} = 1; R = 1.$

$$\sum_{k=0}^{\infty} (-1)^k (2z^2)^k = 1 - 2z^2 + 2^2z^4 - 2^3z^6 + \dots$$

Root test: $\limsup_{k \rightarrow \infty} |a_k|^{\frac{1}{k}} = \limsup_{k \rightarrow \infty} (2^{\frac{k}{2}})^{\frac{1}{k}} = 2^{\frac{1}{2}}; R = 2^{-\frac{1}{2}}.$