Lecture 2: Convergence of Series

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Motivation: functions given by power series

We will define:

$$e^z = 1 + z + \frac{1}{2!} z^2 + \frac{1}{3!} z^3 + \cdots \quad \text{for all } z$$

We will prove:

$$\frac{1}{\sqrt{1-z}} = 1 + \frac{1}{2} z + \frac{1}{2}\frac{3}{2}\frac{1}{2!} z^2 + \frac{1}{2}\frac{3}{2}\frac{5}{2}\frac{1}{3!} z^3 + \cdots \quad \text{for } |z| < 1$$

To start, need to study when infinite sums converge...

Limits of sequences

Definition

If $\{z_n\} = (z_1, z_2, \ldots)$ is a sequence of complex numbers, we say

$$\lim_{n\to\infty} z_n = z \qquad \text{if} \qquad \lim_{n\to\infty} |z_n - z| = 0\,.$$

This is equivalent to the conditions

$$\lim_{n\to\infty} \mathrm{Re}(z_n) = \mathrm{Re}(z) \qquad \text{and} \qquad \lim_{n\to\infty} \mathrm{Im}(z_n) = \mathrm{Im}(z)\,.$$

Example: Consider a complex number z, and let $z_n = z^n$. Then

$$\lim_{n\to\infty} z^n = 0 \qquad \text{if} \qquad |z| < 1\,.$$

Comparison test: If $b_n \geq 0$ and $\lim_{n\to\infty} b_n = 0$, then

$$|z_n - z| \leq b_n \qquad \text{implies} \qquad \lim_{n\to\infty} z_n = z\,.$$

Theorem

Suppose that $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty} w_n = w$. Then

$$\lim_{n\to\infty} (z_n + w_n) = z + w\,, \qquad \lim_{n\to\infty} z_n w_n = zw\,.$$

- First result follows from

$$\big|(z_n + w_n) - (z + w)\big| \le |z_n - z| + |w_n - w|$$

- For second, we use

$$\big|z_n w_n - zw\big| \le |z_n - z|\,|w_n| + |z|\,|w_n - w|$$

To show first term $\to 0$, use $|w_n| \le |w| + 1$ for large n.

Convergence of series

If $(z_0, z_1, z_2, \ldots)$ are complex numbers, we say

$$\sum_{k=0}^{\infty} z_k = z \qquad \text{if} \qquad \lim_{n\to\infty}\left(\sum_{k=0}^{n} z_k\right) = z$$

If $\sum_{k=0}^{\infty} z_k = z$ for some $z \in \mathbb{C}$ we say that $\sum_{k=0}^{\infty} z_k$ **converges.**

Necessary (but not sufficient) condition for convergence:

$$\lim_{k\to\infty} z_k = 0$$

Important example: $z_k = z^k$

$$\sum_{k=0}^{\infty} z^k = (1-z)^{-1} \qquad \text{if} \qquad |z| < 1 .$$

Note: $|z^k| = |z|^k \to 0$ only when $|z| < 1$.

Above is example of an *absolutely convergent* series:

Definition

A series is absolutely convergent if $\sum_{k=1}^{\infty} |z_k|$ converges.

Extremely important fact

An absolutely convergent series is convergent.

Comparison test

Suppose $M_k \geq 0$ are real numbers, and $\sum_{k=0}^{\infty} M_k$ converges.

If $|z_k| \leq M_k$ for every k, then $\sum_{k=0}^{\infty} z_k$ converges absolutely.

Ratio test

If $\lim_{k\to\infty} \frac{|z_{k+1}|}{|z_k|} < 1$, then $\sum_{k=0}^{\infty} z_k$ converges absolutely.

Proof. Choose r such that $\lim_{k\to\infty} \frac{|z_{k+1}|}{|z_k|} < r < 1$.

Then $|z_k| < C\,r^k$ for some C, and $\sum_{k=0}^{\infty} C\,r^k$ converges.

Examples: let z be a complex number

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \tfrac{1}{2!}\, z^2 + \tfrac{1}{3!}\, z^3 + \cdots$$

is convergent for all complex numbers z.

$$\sum_{k=0}^{\infty} \frac{(2k)!\, z^k}{2^{2k}(k!)^2} = 1 + \tfrac{1}{2}\, z + \tfrac{1}{2}\tfrac{3}{2}\, \tfrac{1}{2!}\, z^2 + \tfrac{1}{2}\tfrac{3}{2}\tfrac{5}{2}\, \tfrac{1}{3!}\, z^3 + \cdots$$

is convergent if $|z| < 1$, and not convergent if $|z| > 1$.