Lecture 20: Power series and analytic functions

Hart Smith

Department of Mathematics
University of Washington, Seattle

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Theorem

If the power series $\sum_{k=0}^{\infty} a_k\, z^k$ has radius of convergence R, then the series converges uniformly on $\{z : |z| \le r\}$ for $r < R$.

Proof. Let $f_n(z) = \sum_{k=0}^{n} a_k\, z^k$, and $f(z) = \sum_{k=0}^{\infty} a_k\, z^k$.

$$\big| f(z) - f_n(z) \big| = \left| \sum_{k=n+1}^{\infty} a_k\, z^k \right| \le \sum_{k=n+1}^{\infty} |a_k|\, r^k \quad \text{if } \ |z| \le r.$$

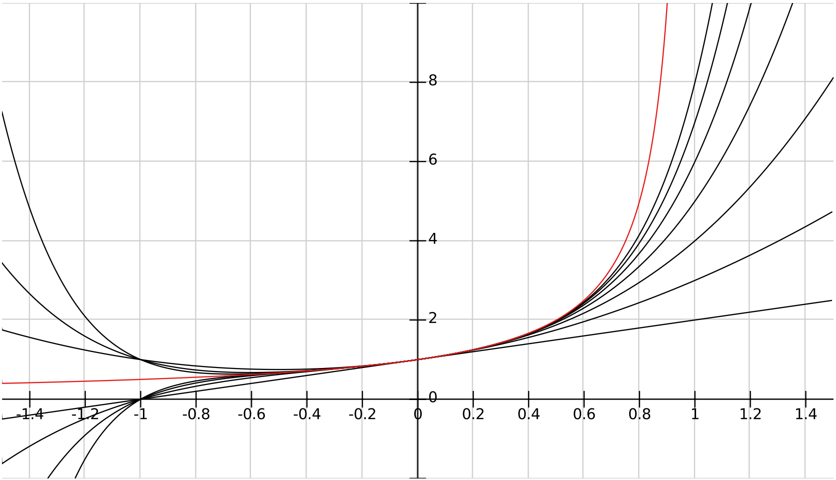
Last sum goes to 0 as $n \to \infty$, since $\sum_{k=0}^{\infty} |a_k|\, r^k$ converges.

Corollary

If the power series $\sum_{k=0}^{\infty} a_k\, z^k$ has radius of convergence R, then

$f(z) = \sum_{k=0}^{\infty} a_k\, z^k$ is a continuous function on $\{z : |z| < R\}$.

First 7 partial sums for $\sum_{k=0}^{\infty} x^k = (1-x)^{-1}$



Application: Suppose $f(z) = \sum_{k=0}^{\infty} a_k z^k$ converges $|z| < R$, and μ is a path contained in $|z| < R$. Then

$$\int_\mu f(z)\,dz \;=\; \sum_{k=0}^{\infty} a_k \int_\mu z^k\,dz$$

In particular, if Δ is a triangle contained in $|z| < R$, then

$$\int_{\partial\Delta} f(z)\,dz \;=\; \sum_{k=0}^{\infty} a_k \int_{\partial\Delta} z^k\,dz \;=\; 0$$

Conclude: $f(z)$ is a continuous function on $\{z : |z| < R\}$,

and $\int_{\partial\Delta} f(z)\,dz \;=\; 0$ for all triangles $\Delta \subset \{z : |z| < R\}$.

Theorem: let R denote the radius of convergence

The function $f(z) = \sum_{k=0}^{\infty} a_k\, z^k$ is analytic on $\{z : |z| < R\}$.

Proof. Consider the function $g(z) \;=\; \sum_{k=1}^{\infty} k\, a_k\, z^{k-1}$.

This is continuous on $E \;=\; \{z : |z| < R\}$, and $\int_{\partial\Delta} g(z)\, dz \;=\; 0$

for all triangles $\Delta \subset E$. Since E is convex, we proved that

$$f(z) \;=\; a_0 + \int_0^z g(w)\, dw \text{ is analytic on } E$$

This also shows: $\left(\sum_{k=0}^{\infty} a_k\, z^k \right)' = \sum_{k=1}^{\infty} k\, a_k\, z^{k-1}$

Higher derivatives

Theorem: let R denote the radius of convergence

The function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ has complex derivatives to every order on $\{z : |z| < R\}$, and $f^{(n)}(z)$ is obtained by differentiating term-by-term:

$$f^{(n)}(z) = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)\, a_k\, z^{k-n}$$

For every n, this series has radius of convergence R.

Important observation: $a_n = \dfrac{f^{(n)}(0)}{n!}$

Example

$$f(z) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{z^k}{k} = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

is analytic on $\{z : |z| < 1\}$, and

$$f(z) = \int_0^z (1 - w + w^2 - w^3 + \cdots)\, dw = \int_0^z \frac{dw}{1+w}$$

This shows that $f(z) = \log(1+z)$ for $|z| < 1$ (principal branch)

If we replace z by $z - 1$, then for $|z-1| < 1$ we have

$$\log(z) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(z-1)^k}{k} = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} \cdots$$

Example

$$f(z) = \sum_{k \text{ odd}}^{\infty} (-1)^{\frac{k-1}{2}} \, \frac{z^k}{k} = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots$$

is analytic on $\{z : |z| < 1\}$, and

$$f(z) = \int_0^z (1 - w^2 + w^4 - w^6 + \cdots) \, dw = \int_0^z \frac{dw}{1 + w^2}$$

This shows that $f(x) = \arctan(x)$ for real $x \in (-1, 1)$.

- The function $\arctan(x)$ is well behaved for all $x \in \mathbb{R}$, but $f(z)$ blows up as $z \to \pm i$.
- This explains why $R = 1$ for the power series.