Lecture 21: Power series expansions of analytic functions

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Recall from Lecture 18

Index function for γ the circle of radius r

Let γ trace $\partial D_r(0)$: $\quad \gamma(t) = re^{it}$, $t \in [0, 2\pi]$.

Then: $$\mathrm{ind}_\gamma(z) = \begin{cases} 1\,, & |z| < r\,, \\ 0\,, & |z| > r\,. \end{cases}$$

Cauchy integral formula for the circle

Assume $f(z)$ analytic on an open set E containing $\{z : |z| \leq r\}$.

Then, for $|z| < r$: $$f(z) = \frac{1}{2\pi i} \int_{|w|=r} \frac{f(w)}{w - z}\, dw$$

Suppose $|z| < r\,,\ |w| = r$. Then $\left|\dfrac{z}{w}\right| < 1,$ so we can write

$$\frac{1}{w-z} = \frac{1}{w}\,\frac{1}{1-\dfrac{z}{w}} = \frac{1}{w}\sum_{k=0}^{\infty}\Big(\frac{z}{w}\Big)^{k} = \sum_{k=0}^{\infty}\frac{z^k}{w^{k+1}}$$

This gives, for each $|z| < r$,

$$f(z) = \frac{1}{2\pi i}\int_{|w|=r}\ \sum_{k=0}^{\infty}\frac{z^k}{w^{k+1}}\,f(w)\,dw$$

Claim: we can bring summation outside the integral, to get

$$f(z) = \sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\int_{|w|=r}\frac{f(w)}{w^{k+1}}\,dw\right)\,z^k = \sum_{k=0}^{\infty} a_k\,z^k$$

Proof: The series expansion: $$(1-\lambda)^{-1} = \sum_{k=0}^{\infty} \lambda^k$$

converges uniformly on the set $\{|\lambda| \leq c\}$, if $c < 1$.

It follows that, for each fixed z with $|z| < r$, the expansion

$$\frac{1}{w-z} = \frac{1}{w} \frac{1}{1 - \frac{z}{w}} = \frac{1}{w} \sum_{k=0}^{\infty} \left(\frac{z}{w}\right)^k = \sum_{k=0}^{\infty} \frac{z^k}{w^{k+1}}$$

converges uniformly on the set $\{w : |w| = r\}$ since

$$\left|\frac{z}{w}\right| = c, \quad \text{where} \quad c = \frac{|z|}{r} < 1.$$

Theorem

If $f(z)$ is analytic on an open set containing $\overline{D_r}(0)$, then $f(z)$ has a convergent power series expansion on $D_r(0)$,

$$f(z) = \sum_{k=0}^{\infty} a_k\, z^k\,, \quad \text{where} \quad a_k \;=\; \frac{1}{2\pi i}\int_{|w|=r} \frac{f(w)}{w^{k+1}}\, dw$$

Expansions about general points

If $f(z)$ is analytic on an open set containing $\overline{D_r}(z_0)$, then $f(z)$ has a convergent power series expansion for $z \in D_r(z_0)$,

$$f(z) = \sum_{k=0}^{\infty} a_k\,(z - z_0)^k, \quad a_k \;=\; \frac{1}{2\pi i}\int_{|w-z_0|=r} \frac{f(w)}{(w - z_0)^{k+1}}\, dw$$

- If $f(z)$ is analytic on E, and $z_0 \in E$, this shows that $f(z)$ is equal to its Taylor expansion on $D_r(z_0)$:

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k$$

  for the largest r such that $D_r(z_0) \subset E$, and $r = \infty$ if $E = \mathbb{C}$.

  Remark: The power series expansion of f may converge on a larger set than the largest $D_r(z_0)$ contained in E.

- If $f(z)$ is analytic on E, then $f(z)$ has complex derivatives to all orders on E; in particular $f'(z)$ is analytic.

  Proof. For each $z_0 \in E$, $f(z)$ is a convergent power series on some $D_r(z_0)$, so has derivatives of all order on $D_r(z_0)$.

Examples

The function $\dfrac{1}{1+z^2}$ is analytic on $\mathbb{C} \setminus \{i, -i\}$.

- Its Taylor expansion about $z_0 = 0$ converges on $D_1(0)$.
- Its Taylor expansion about $z_0 = 10$ converges on $D_{\sqrt{101}}(10)$.

The function $\tan(z)$ is analytic on $\mathbb{C} \setminus \{k\pi,\, k \in \mathbb{Z}\}$.

- Its Taylor expansion about $z_0 = 0$ converges on $D_\pi(0)$.
- Its Taylor expansion about $z_0 = i$ converges on $D_{\sqrt{1+\pi^2}}(i)$.

Fact: The power series expansion of $\log z$ about z_0 has radius of convergence $R = |z_0|$, for $z_0 \neq 0$, and any branch of $\log z$.

Proof. The radius of convergence for $\log z$ about z_0 is the same as the radius of convergence for its derivative, $(\log z)' = z^{-1}$. The function z^{-1} is analytic on $\mathbb{C} \setminus \{0\} \supset D_{|z_0|}(z_0)$.

- For any branch of $\log z$, its power series expansion at z_0 is

$$\log(z_0) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{(z - z_0)^k}{z_0^k}$$

- If $D_{|z_0|}(z_0)$ extends across the cut line, the expansion does not agree with that branch across the cut line.