Lecture 22: Power series and Analytic functions

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Power series $\Rightarrow$ analytic function: if $R =$ radius convergence,

The function $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ is analytic on $D_R(z_0)$,

and $f'(z) = \sum_{k=1}^{\infty} k\, a_k (z - z_0)^{k-1}$.

Consequence: if $f'(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k$ on $D_R(z_0)$, then

$$f(z) = f(z_0) + \sum_{k=1}^{\infty} \frac{b_{k-1}}{k} (z - z_0)^k , \quad z \in D_R(z_0)$$

- radius convergence of $f(z)$ = radius convergence of $f'(z)$

Example: $f(z) = \log z\,, \qquad f'(z) = \dfrac{1}{z}$

Expansion of z^{-1} on $|z-1| < 1$:

$$\frac{1}{z} = \frac{1}{1+(z-1)} = \sum_{k=0}^{\infty} (-1)^k \, (z-1)^k$$

Expansion of $\log z$ (principal branch) on $|z-1| < 1$:

$$\log z = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \, (z-1)^k$$

Example: $f(z) = \arctan z\,, \qquad f'(z) = \dfrac{1}{1+z^2}$

Expansion of $\dfrac{1}{1+z^2}$ on $|z| < 1$:

$$\frac{1}{1+z^2} = \sum_{k=0}^{\infty} (-1)^k\, z^{2k}$$

Expansion of $\arctan z$ (principal branch) on $|z| < 1$:

$$\arctan z = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\, z^{2k+1}$$

Power series expansions of analytic functions

If $f(z)$ is analytic on an open set containing $D_r(z_0)$, then $f(z)$ has a convergent power series expansion for $z \in D_r(z_0)$,

$$f(z) = \sum_{k=0}^{\infty} a_k \, (z - z_0)^k, \qquad a_k = \frac{f^{(k)}(z_0)}{k!}$$

Remark: $D_r(z_0)$ is an open set containing $D_r(z_0)$.

Remark: The power series expansion of f may converge on a larger set than the largest $D_r(z_0)$ contained in E.

Fact: The power series expansion of $\log z$ about z_0 has radius of convergence $R = |z_0|$, for $z_0 \neq 0$, and any branch of $\log z$.

Proof. The radius of convergence for $\log z$ about z_0 is the same as the radius of convergence for its derivative, $(\log z)' = z^{-1}$. The function z^{-1} is analytic on $\mathbb{C} \setminus \{0\} \supset D_{|z_0|}(z_0)$.

- For any branch of $\log z$, its power series expansion at z_0 is

$$\log(z_0) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{(z - z_0)^k}{z_0^k}$$

- If $D_{|z_0|}(z_0)$ extends across the cut line, the series expansion does not agree with that branch across the cut line.

Theorem: Cauchy's estimates

If $f(z)$ is analytic on an open set containing $\overline{D_r}(z_0)$, then

$$\frac{|f^{(k)}(z_0)|}{k!} \leq \frac{M}{r^k}, \qquad M = \max_{|w-z_0|=r} |f(w)|$$

Proof. The coefficients $a_k = f^{(k)}(z_0)/k!$ are given by

$$a_k = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{(w-z_0)^{k+1}}\, dz$$

Use Theorem 2.4.9: $\ell(\gamma) = 2\pi r$, $|f(w)/(w-z_0)^{k+1}| \leq M/r^{k+1}$

Remark. This proves that $\sum_{k=0}^{\infty} a_k (z-z_0)^k$ converges on $D_r(z_0)$;

we already knew this from the proof that it equals $f(z)$ there.