Lecture 23: Liouville's Theorem, The Fundamental Theorem of Algebra

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Math 427, Autumn 2019

Theorem: Cauchy's estimates

If $f(z)$ is analytic on an open set containing $\overline{D_r}(z_0)$, then

$$\frac{|f^{(k)}(z_0)|}{k!} \leq \frac{M}{r^k}, \qquad M = \max_{|z-z_0|=r} |f(z)|$$

Proof. The coefficients $a_k = f^{(k)}(z_0)/k!$ are given by

$$a_k = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{k+1}}\, dz$$

Use Theorem 2.4.9: $\ell(\gamma) = 2\pi r$, $|f(z)/(z-z_0)^{k+1}| \leq M/r^{k+1}$.

Remark. This proves that $\sum_{k=0}^{\infty} a_k (z-z_0)^k$ converges on $D_r(z_0)$;

we already knew this from the proof that it equals $f(z)$ there.

Bounded entire functions

Liouville's Theorem

Suppose $f(z)$ is an entire function; that is, it is analytic on $\mathbb{C}$.
If $|f(z)| \leq M$ for all $z \in \mathbb{C}$, for some M, then f is constant.

Proof. It suffices to show that $f'(z_0) = 0$ for all $z_0 \in \mathbb{C}$.

Apply Cauchy's estimate: for every $r > 0$,

$$|f'(z_0)| \leq \frac{1}{r} \sup_{|z-z_0|=r} |f(z)| \leq \frac{M}{r}$$

Letting $r \to \infty$ shows that $|f'(z_0)| = 0$.

Entire functions of polynomial growth

Theorem

Suppose $f(z)$ is an entire function, and $|f(z)| \leq A + B|z|^n$, for some real numbers A and B. Then $f(z) = \sum_{k=0}^{n} a_k\, z^k$, some a_k.

Proof. It suffices to show that $f^{(k)}(0) = 0$ for all $k > n$, since the Taylor expansion of $f(z)$ about $z_0 = 0$ equals $f(z)$ for all $z \in \mathbb{C}$.

Apply Cauchy's estimate: for every $r > 0$,

$$|f^{(k)}(0)| \leq \frac{k!}{r^k} \sup_{|z|=r} |f(z)| \leq \frac{k!\,(A + B\,r^n)}{r^k}$$

Letting $r \to \infty$ shows that $|f^{(k)}(0)| = 0$ when $k > n$.

Lower bound estimates for polynomials

Lemma

Suppose $f(z) = \sum_{k=0}^{n} a_k\, z^k$, where $a_n \neq 0$. Then, for some r_0,

$$|f(z)| \;\geq\; \tfrac{1}{2}|a_n|\,|z|^n\,, \quad \text{if} \quad |z| \geq r_0\,.$$

Proof. By the triangle inequality, we have

$$|f(z)| \;\geq\; |a_n z^n| - \Big(|a_{n-1}\, z^{n-1}| + \;\cdots\; + |a_1\, z| + |a_0|\Big)\,.$$

Take r_0 so that $r_0 \geq 2n\,|a_k|/|a_n|$ for all k. Then if $|z| \geq r_0$

$$|a_k||z|^k \;\leq\; \frac{|a_k||z|^{k+1}}{r_0} \;\leq\; \frac{|a_n||z|^n}{2n} \quad \text{if} \;\; k < n\,.$$

Together with the previous inequality we get $|f(z)| \;\geq\; \frac{1}{2}|a_n|\,|z|^n$.

Theorem

Suppose that $p(z) = \sum_{k=0}^{n} a_k\, z^k$, where $n \geq 1$ and $a_n \neq 0$. Then $p(z) = 0$ for some $z \in \mathbb{C}$.

Proof. Proof by contradiction. Suppose $p(z) \neq 0$ for all z. Then

$$\frac{1}{p(z)} \text{ is analytic on } \mathbb{C}, \text{ and } \left|\frac{1}{p(z)}\right| \leq \frac{2}{|a_n|\, r^n} \text{ if } |z| \geq r_0\,.$$

Liouville's Theorem implies $p(z)^{-1}$ is constant, a contradiction.

The Fundamental Theorem of Algebra

Every polynomial $p(z) = \sum_{k=0}^{n} a_k\, z^k$ can be factored over $\mathbb{C}$,

$$p(z) = a_n(z - z_n) \cdots (z - z_2)(z - z_1) \text{ where } z_j \in \mathbb{C}.$$

Proof. From algebra: if $p(z_n) = 0$, then $p(z) = (z - z_n)q(z)$, where $q(z)$ is a polynomial of order $n - 1$.