Lecture 24: Zeroes of analytic functions

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Assume $f(z)$ analytic on $E \subset \mathbb{C}$, and $f(z_0) = 0$. If $|z - z_0| < R$:

$$f(z) = \sum_{k=1}^{\infty} a_k (z - z_0)^k, \qquad a_k = \frac{f^{(k)}(z_0)}{k!}$$

Two possibilities:

- If $a_k = 0$ for every k, then $f(z) = 0$ for $|z - z_0| < R$.
- If for some m, we have $a_m \neq 0$ but $a_k = 0$ when $k < m$, we say that $f(z)$ has a zero of order m at z_0. Equivalently:
- $f(z)$ has a zero of order m at z_0 if:

$$f^{(m)}(z_0) \neq 0, \quad \text{and} \quad f^{(k)}(z_0) = 0 \text{ for } k < m.$$

Examples

- $\sin(z)$ has a zero of order 1 at $z_0 = 0$:

$$\sin(z) = z - \frac{1}{3!}z^3 + \cdots \quad , \qquad \text{so} \;\; a_0 = 0\,, \;\; a_1 \neq 0\,.$$

Can also check: $\sin(z) = 0$, $\sin'(0) = \cos(0) = 1 \neq 0\,.$

- $\sin(z)$ has a zero of order 1 at $z_0 = k\pi$.

- $z^3 - 1$ has a zero of order 1 at $z_0 = 1$:

$$z^3 - 1 = 0 \quad \text{when} \quad z = 1, \quad (z^3 - 1)' = 3z^2 = 3 \quad \text{when} \quad z = 1.$$

- $e^z - z - 1$ has a zero of order 2 at $z_0 = 0$:

$$e^z - z - 1 \;=\; \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots$$

Theorem: assume f analytic on $E \subset \mathbb{C}$

If $f(z)$ has a zero of order m at z_0, there is $g(z)$ analytic on E:

$$f(z) = (z - z_0)^m g(z) , \quad \text{where} \quad g(z_0) \neq 0 .$$

Proof. For $|z - z_0| < R$ we can write:

$$f(z) = \sum_{k=m}^{\infty} a_k (z - z_0)^k = (z - z_0)^m \sum_{k=0}^{\infty} a_{k+m} (z - z_0)^k .$$

$$\text{Define:} \qquad g(z) = \begin{cases} \displaystyle\sum_{k=0}^{\infty} a_{k+m} (z - z_0)^k , & |z - z_0| < R , \\ \dfrac{f(z)}{(z - z_0)^m} , & z \neq z_0 . \end{cases}$$

If $f(z)$ has a zero of order m at z_0, then $\dfrac{f(z)}{(z - z_0)^m}$, defined on the set $E \setminus \{z_0\}$, extends to an analytic function on E.

Theorem: L'Hôpital's rule

If $f(z_0) = g(z_0) = 0$, then $\displaystyle\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{f'(z)}{g'(z)}$

Proof. Unless the order of zeroes of f and g at z_0 are the same, then both limits are either 0 or ∞. If f and g have zero order m:

$$f(z) = \sum_{k=m}^{\infty} a_k \,(z - z_0)^k\,, \qquad g(z) = \sum_{k=m}^{\infty} b_k \,(z - z_0)^k\,,$$

$$f'(z) = \sum_{k=m}^{\infty} k\,a_k (z - z_0)^{k-1}\,, \qquad g'(z) = \sum_{k=m}^{\infty} k\,b_k \,(z - z_0)^{k-1}\,,$$

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{a_m}{b_m}\,, \qquad \lim_{z\to z_0} \frac{f'(z)}{g'(z)} = \frac{m a_m}{m b_m} = \frac{a_m}{b_m}$$

Zeroes of analytic functions are isolated

Theorem: suppose f is analytic on connected open set $E \subset \mathbb{C}$.

If $f(z_0) = 0$, and f is not identically 0, then for some $r > 0$:

$$f(z) \neq 0 \quad \text{if} \quad 0 < |z - z_0| < r\,.$$

Proof. Write $f(z) = (z - z_0)^m h(z)\,,\quad h(z_0) \neq 0\,.$ By continuity:

$$h(z) \neq 0 \quad \text{if} \quad |z - z_0| < r\,, \quad \text{for some} \ \ r > 0\,.$$

Application. If $\{z_k\} \subset E$ is a sequence with $\lim_{k\to\infty} z_k = z_0 \in E$, and $f(z_k) = g(z_k)$ for all k, then $f(z) = g(z)$ on E.

Proof. Let $h(z) = f(z) - g(z)$. Then $h(z_0) = 0$ by continuity. For every $r > 0$, is some z_k with $|z_k - z_0| < r$, and $h(z_k) = 0$, so $h(z)$ must be identically 0.