Lecture 25: The structure of isolated singularities

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Definition: Suppose $f(z)$ is defined for $0 < |z - z_0| < r$.

We say that $\lim_{z\to z_0} f(z) = \infty$ if, for all M, there is $\delta > 0$ so that

$$|f(z)| > M \quad \text{if} \quad 0 < |z - z_0| < \delta.$$

Examples: $\displaystyle \lim_{z\to 0} \frac{1}{z} = \infty\,, \qquad \lim_{z\to \pi} \tan(z) = \infty\,.$

- If $h(z)$ is analytic on $D_r(z_0)$ and $h(z_0) \neq 0$, then when $m \geq 1$

$$\lim_{z\to z_0} \frac{h(z)}{(z - z_0)^m} = \infty\,.$$

Important fact: If $\displaystyle \lim_{z\to z_0} f(z) = \infty$ then $\displaystyle \lim_{z\to z_0} \frac{1}{f(z)} = 0\,.$

The function $g(z) = \begin{cases} 1/f(z)\,, & z \neq z_0 \\ 0\,, & z = z_0 \end{cases}$ is continuous at z_0.

Classification of isolated singularities

Definition: assume f is analytic on open set $E \subset \mathbb{C}$ and $z_0 \notin E$.

f has an isolated singularity at z_0 if, for some $r > 0$, E contains the punctured disc $D_r(z_0) \setminus \{z_0\} = \{z : 0 < |z - z_0| < r\}$.

Isolated singularities are classified as one of 3 types:

- f has a *removable singularity* at z_0 if $f(z)$ is bounded on some punctured disc about z_0:

  $$|f(z)| \leq M \quad \text{when} \quad 0 < |z - z_0| < r', \;\; \text{some } M, r' > 0.$$

- f has a *pole* at z_0 if $\lim_{z \to z_0} f(z) = \infty$.

- Everything else: f has an *essential singularity* at z_0.

Removable singularities

Theorem: assume f analytic on $D_r(z_0) \setminus \{z_0\}$

If f has a removable singularity at z_0, then f equals a function that is analytic function on all of $D_r(z_0)$.

Proof. The function $g(z) = \begin{cases} (z - z_0) f(z), & z \neq z_0\,, \\ \quad 0, & z = z_0\,, \end{cases}$

is analytic on $D_r(z_0) \setminus \{z_0\},$ and continuous on $D_r(z_0)$.

- By Lecture 15: $g(z)$ is analytic on $D_r(z_0)$.
- Since $g(z_0) = 0,$ we can write $g(z) = (z - z_0) h(z)$, where $h(z)$ is analytic on $D_r(z_0)$.
- $f(z) = h(z)$ on $D_r(z_0) \setminus \{z_0\},$ and $h(z)$ is analytic on $D_r(z_0)$.

Poles

Theorem: assume $f(z)$ analytic on $D_r(z_0) \setminus \{z_0\}$

If f has a pole at z_0, then $f(z) = \dfrac{h(z)}{(z-z_0)^m}$ for some m, where $h(z)$ is an analytic function on $D_r(z_0)$, and $h(z_0) \neq 0$.

Proof. Define $g(z) = \begin{cases} 1/f(z)\,, & z \neq z_0 \\ 0\,, & z = z_0 \end{cases}$

- $g(z)$ has a removable singularity at z_0, so $g(z)$ is analytic.
- Write $g(z) = (z-z_0)^m \tilde{g}(z)$, with $\tilde{g}(z_0) \neq 0$.
- Write $f(z) = \dfrac{1}{g(z)} = \dfrac{h(z)}{(z-z_0)^m}$, where $h(z) = \dfrac{1}{\tilde{g}(z)}$

- If $f(z) = \dfrac{h(z)}{(z-z_0)^m}$, $h(z)$ analytic on $D_r(z_0)$ and $h(z_0) \neq 0$, we say $f(z)$ has a *pole of order* m at z_0. (Note: m is unique.)

Equivalent definition: f has pole order m at z_0 if:

$$f(z) = \sum_{k=-m}^{\infty} a_k\,(z-z_0)^k\,, \quad z \in D_r(z_0) \setminus \{z_0\}\,, \quad a_{-m} \neq 0$$

where $\displaystyle\sum_{k=0}^{\infty} a_k\,(z-z_0)^k$ converges on $D_r(z_0)$.

The term $\displaystyle\sum_{k=-m}^{-1} a_k\,(z-z_0)^k = \frac{a_{-m}}{(z-z_0)^m} + \cdots + \frac{a_{-1}}{(z-z_0)}$

is called the **principal part** of f at z_0.