

Lecture 26: Analytic functions with poles

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If $f(z) = \frac{h(z)}{(z - z_0)^m}$, $h(z)$ analytic on $D_r(z_0)$ and $h(z_0) \neq 0$,

we say $f(z)$ has a **pole of order m** at z_0 . We can then write:

$$f(z) = \sum_{k=-m}^{\infty} a_k (z - z_0)^k, \quad z \in D_r(z_0) \setminus \{z_0\}$$

where $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ converges on $D_r(z_0)$ for some $r > 0$.

The term $\sum_{k=-m}^{-1} a_k (z - z_0)^k = \frac{a_{-m}}{(z - z_0)^m} + \cdots + \frac{a_{-1}}{(z - z_0)}$

is called the **principal part** of $f(z)$ at z_0 .

Easy fact: If $g(z)$ has a zero of order m at z_0 , and $f(z_0) \neq 0$, then $f(z)/g(z)$ has a pole of order m at z_0 .

Proof. Write $g(z) = (z - z_0)^m h(z)$, where $h(z) \neq 0$ on $D_r(z_0)$.

Then:
$$\frac{f(z)}{g(z)} = \frac{f(z)h(z)^{-1}}{(z - z_0)^m}$$

- $\tan(z)$ has a pole of order 1 at $z = (k + \frac{1}{2})\pi$, $k \in \mathbb{Z}$.
- $\csc(z) = \sin(z)^{-1}$ has a pole of order 1 at $z = k\pi$.
- $(\sin(z) - z)^{-1}$ has a pole of order 3 at $z = 0$.
- $(z^3 - 1)^{-1}$ has a pole of order 1 at $z = e^{i2\pi k/3}$, $k = 0, 1, 2$.

Calculating the principal part for simple poles: $m = 1$

Observation

Suppose that $f(z)$ has a simple pole at z_0 :

$$f(z) = \frac{a_{-1}}{z - z_0} + \sum_{k=0}^{\infty} a_k (z - z_0)^k, \quad 0 < |z - z_0| < r.$$

Then $a_{-1} = \lim_{z \rightarrow z_0} (z - z_0) f(z)$.

Example. To find the principal part of $(\sin z)^{-1}$ at $z_0 = k\pi$:

$$a_{-1} = \lim_{z \rightarrow k\pi} \frac{z - k\pi}{\sin z} = \lim_{z \rightarrow k\pi} \frac{1}{\sin' z} = \frac{1}{\cos(k\pi)} = (-1)^k$$

Principal part at $z = k\pi$ is: $\frac{(-1)^k}{z - k\pi}$

Example. Find the principal part of $\tan z$ at $z_0 = \frac{\pi}{2}$.

Solution:
$$\lim_{z \rightarrow \frac{\pi}{2}} (z - \frac{\pi}{2}) \tan z = \frac{\lim_{z \rightarrow \frac{\pi}{2}} \sin z}{\lim_{z \rightarrow \frac{\pi}{2}} \left(\frac{\cos z}{z - \frac{\pi}{2}} \right)} = \frac{\sin \frac{\pi}{2}}{\cos' \frac{\pi}{2}} = -1.$$

Principal part is:
$$\frac{-1}{z - \frac{\pi}{2}}$$

Example. Find the principal part of $(z^2 + 1)^{-1}$ at $z_0 = i$.

Solution:
$$\lim_{z \rightarrow i} \frac{z - i}{z^2 + 1} = \lim_{z \rightarrow i} \frac{1}{z + i} = \frac{1}{2i} = -\frac{1}{2}i.$$

Principal part is:
$$\frac{-\frac{1}{2}i}{z - i}$$

Partial fraction decomposition:
$$\frac{1}{z^2 + 1} = \frac{-\frac{1}{2}i}{z - i} + \frac{\frac{1}{2}i}{z + i}$$

Partial fraction decomposition

Theorem

Suppose that $q(z) = (z - z_1)^{m_1} (z - z_2)^{m_2} \cdots (z - z_n)^{m_n}$

Let $p_{z_j}(z) = \sum_{k=-m_j}^{-1} \frac{a_k}{(z - z_j)^k}$ be the principal part of $\frac{1}{q(z)}$ at z_j .

Then $\frac{1}{q(z)} = \sum_{j=1}^n p_{z_j}(z)$.

Proof. $\frac{1}{q(z)} - \sum_{j=1}^n p_{z_j}(z)$ is analytic on $\mathbb{C} \setminus \{z_1, \dots, z_n\}$, and analytic at each z_k , since $\frac{1}{q(z)} - p_{z_k}(z)$ and p_{z_j} for $j \neq k$ are.

$\lim_{z \rightarrow \infty} \frac{1}{q(z)} - \sum_{j=1}^n p_{z_j}(z) = 0$, so by Liouville's theorem it $= 0$.