Lecture 28: The maximum modulus theorem

Hart Smith

Department of Mathematics
University of Washington, Seattle

Math 427, Autumn 2019

Theorem

Assume that $f(t)$ is a continuous, real valued function on $[a, b]$, and $f(t) \leq M$ for all $t \in [a, b]$. Then if $\dfrac{1}{b-a}\displaystyle\int_a^b f(t)\,dt = M$, we must have $f(t) = M$ for all $t \in [a, b]$.

Corollary

Assume $f(t)$ is a continuous, complex valued function on $[a, b]$, and let $c = \dfrac{1}{b-a}\displaystyle\int_a^b f(t)\,dt$. Then if $|f(t)| \leq |c|$ for all $t \in [a, b]$, we must have $f(t) = c$ for all $t \in [a, b]$.

Proof. Write $c^{-1}f(t) = g(t) + ih(t)$, so $|g(t)| \leq 1$ for all $t \in [a, b]$. Then: $1 = \dfrac{1}{b-a}\displaystyle\int_a^b g(t) + ih(t)\ dt$, so $g(t) = 1$ for all t, which means $h(t) = 0$ for all t, since $\sqrt{g(t)^2 + h(t)^2} \leq 1$ for all t. $\square$

Theorem: suppose $f(z)$ is analytic on $D_r(z_0)$

If $|f(z)| \leq |f(z_0)|$ for all $z \in D_r(z_0)$, then $f(z) = f(z_0)$ on $D_r(z_0)$.

Proof. For any $r' < r$, by the Cauchy integral formula

$$f(z_0) = \frac{1}{2\pi i} \int_{|w-z_0|=r'} \frac{f(w)}{w - z_0}\, dw = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r'e^{it})\, dt$$

By the Corollary, $f(z_0 + r'e^{it}) = f(z_0)$ for all t. Since this holds for any $r' < r$, we have $f(z) = f(z_0)$ for all $z \in D_r(z_0)$.

Maximum modulus theorem

Assume $f(z)$ is analytic on E, and continuous on $\bar{E}$, where E is a bounded, connected, open set. Then the maximum of $|f(z)|$ on $\bar{E}$ occurs on ∂E (and only on ∂E if f is not constant).

Proof. $|f(z)|$ can't equal $\max_{\bar{E}} |f|$ for $z \in E$ unless f is constant.

Examples

- $|z^4 - 1|$ on $\{z : |z| \le 1\}$. Maximum is at some $z = e^{i\theta}$,

$$\left|e^{4i\theta} - 1\right|^2 = \left(\cos(4\theta) - 1\right)^2 + \sin(4\theta)^2$$

Setting the derivative equal to 0 gives $\sin(4\theta) = 0$:

$$\theta = 0, \pm\tfrac{\pi}{2}, \pi \ \text{(minimum)}, \qquad \theta = \pm\tfrac{\pi}{4}, \pm\tfrac{3\pi}{4} \ \text{(maximum)}$$

- $|e^z|$ on the square: $-1 \le \mathrm{Re}(z) \le 1$, $-1 \le \mathrm{Im}(z) \le 1$.

$$|e^{x+iy}| = e^x \quad \text{does not depend on } y.$$

e^x is maximized over $-1 \le x \le 1$ at $x = 1$, so maximum is achieved at every point on right edge of the square.

Mean value property for harmonic functions

Theorem

Suppose u is harmonic on E. Then, whenever $\overline{D_r}(z_0) \subset E$,

$$u(z_0) \;=\; \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta$$

Proof. There is analytic f on $\overline{D_r}(z_0) \subset E$, with $u(z) = \mathrm{Re}(f(z))$,

$$u(z_0) = \mathrm{Re}\left(\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta\right) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta$$

Applying our first Theorem to $u(z)$ gives us

Maximum principle for harmonic functions

Assume u is harmonic on E, and continuous on $\bar{E}$, where E is a bounded, connected, open set. Then the maximum of u on $\bar{E}$ occurs on ∂E (and only on ∂E if u is not constant).