

# Lecture 3: The exponential function

Hart Smith

Department of Mathematics  
University of Washington, Seattle

Math 427, Autumn 2019

## Definition

For a complex number  $z$ , we define

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{1}{2!} z^2 + \frac{1}{3!} z^3 + \dots$$

Some immediate facts:

- $e^0 = 1$
- $e^x$  is a real number if  $x$  is real.
- $\overline{e^z} = e^{\bar{z}}$

**Fundamental property:**  $e^{w+z} = e^w e^z$

**Proof.** By definition:  $e^{w+z} = \sum_{n=0}^{\infty} \frac{(w+z)^n}{n!}$

Use the binomial theorem:  $(w+z)^n = \sum_{j+k=n} \frac{n!}{j!k!} w^j z^k$

$$e^{w+z} = \sum_{n=0}^{\infty} \sum_{j+k=n} \frac{w^j}{j!} \frac{z^k}{k!}$$

This is a re-ordering of the sum

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{w^j}{j!} \frac{z^k}{k!} = e^w e^z$$

1	$w + z$	$\frac{1}{2!}(w + z)^2$	$\frac{1}{3!}(w + z)^3$	$\frac{1}{4!}(w + z)^4$	...
1	$z$	$\frac{1}{2!}z^2$	$\frac{1}{3!}z^3$	$\frac{1}{4!}z^4$	...
$w$	$wz$	$\frac{1}{2!}wz^2$	$\frac{1}{3!}wz^3$	$\frac{1}{4!}wz^4$	...
$\frac{1}{2!}w^2$	$\frac{1}{2!}w^2z$	$\frac{1}{2!2!}w^2z^2$	$\frac{1}{2!3!}w^2z^3$	$\frac{1}{2!4!}w^2z^4$	...
$\frac{1}{3!}w^3$	$\frac{1}{3!}w^3z$	$\frac{1}{3!2!}w^3z^2$	$\frac{1}{3!3!}w^3z^3$	$\frac{1}{3!4!}w^3z^4$	...
$\frac{1}{4!}w^4$	$\frac{1}{4!}w^4z$	$\frac{1}{4!2!}w^4z^2$	$\frac{1}{4!3!}w^4z^3$	$\frac{1}{4!4!}w^4z^4$	...
:	:	:	:	:	..

## Additional properties

- $e^{-z} = (e^z)^{-1}$  (since  $e^{-z}e^z = e^0 = 1$ )
- $|e^{iy}| = 1$  for  $y$  real, ( $\overline{e^{iy}} = e^{-iy}$ , so  $e^{iy}\overline{e^{iy}} = |e^{iy}|^2 = 1$ )
- $e^z = e^{\operatorname{Re}(z)}e^{i\operatorname{Im}(z)}$ , so  $|e^z| = e^{\operatorname{Re}(z)}$
- $e^{iy} = \cos y + i \sin y$

$$\begin{aligned} e^{iy} &= 1 + (iy) + \frac{1}{2!}(iy)^2 + \frac{1}{3!}(iy)^3 + \frac{1}{4!}(iy)^4 + \frac{1}{5!}(iy)^5 + \dots \\ &= 1 - \frac{1}{2!}y^2 + \frac{1}{4!}y^4 + \dots \\ &\quad + iy - i\frac{1}{3!}y^3 + i\frac{1}{5!}y^5 \end{aligned}$$

# Consequences of $e^{x+iy} = e^x(\cos y + i \sin y)$

## Theorem

$e^{z+2\pi i} = e^z$ , and if  $e^w = e^z$  then  $w = z + k2\pi i$  for some  $k$ .

**Proof.** Let  $z = x + iy$ ,  $w = u + iv$ , and assume  $e^z = e^w$ .

- $u = x$  since  $e^u = e^x$ .
- $v = y + k2\pi$  since  $\cos v + i \sin v = \cos y + i \sin y$ .

## Polar decomposition of $z \in \mathbb{C}$

If  $z \neq 0$ , then  $z = r e^{i\theta}$  for some  $r > 0$  and some real  $\theta$ .

- $r = |z|$  is determined, choose  $\theta$  from polar coordinates.
- If  $z = r e^{i\theta'}$ , then  $\theta' = \theta + k2\pi$ .