Lecture 4: Trig and n-th root functions

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For a complex number z, we define

$$\cos z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} = 1 - \frac{1}{2!} z^2 + \frac{1}{4!} z^4 - \cdots$$

$$\sin z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} = z - \frac{1}{3!} z^3 + \frac{1}{5!} z^5 - \cdots$$

From the relation $e^{iz} = \cos z + i \sin z$ we can write

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

- $\cos(z + 2\pi) = \cos z\,, \qquad \sin(z + 2\pi) = \sin z\,.$

Fact

The zeroes of $\sin z$ and $\cos z$ are all real numbers:

$$\begin{cases} \sin z = 0 \quad \text{only if} \quad z = k\pi \\ \cos z = 0 \quad \text{only if} \quad z = \frac{\pi}{2} + k\pi \end{cases} \qquad \text{for an integer } k.$$

Proof. $\cos z = 0$ is equivalent to

$$e^{iz} = -e^{-iz} \qquad \text{hence} \qquad e^{2iz} = -1\,.$$

By last lecture, this implies $2iz = \pi i + k\,2\pi i$ some integer k.

- $\tan z = \dfrac{\sin z}{\cos z}$ is defined for $z \neq \frac{\pi}{2} + k\pi$.

Polar form and the argument of z

Polar form for $z \neq 0$: $\quad z = r\,e^{i\theta}\,, \quad r = |z|$.

θ is called *the argument* of z, we write $\theta = \arg(z)$.

- $\arg(z)$ is *multi-valued function*: $\theta + k\,2\pi$ also works.
- Unique value in fixed interval length 2π : e.g. $-\pi < \theta \leq \pi$.

Behavior under multiplication:

$$z = r_1\,e^{i\theta_1}, \quad w = r_2 e^{i\theta_2}, \quad \text{then} \quad zw = r_1\,r_2\,e^{i(\theta_1+\theta_2)}$$

Works for any choice of $\arg(w)$ and $\arg(z)$.

n-th roots of complex numbers

To solve $w^n = z$, for positive integer n, and $z \neq 0$:

- Write $z = re^{i\theta}$. Necessarily $w = r^{\frac{1}{n}} e^{i\theta'}$, where $e^{in\theta'} = e^{i\theta}$,

$$n\theta' = \theta + 2\pi k \quad \text{for some integer } k$$

- Obtain exactly n distinct solutions w, by taking

$$\theta' = \frac{\theta}{n} + \frac{2\pi k}{n} \quad \text{for } k = 0 \ldots, n-1 .$$

Example: $w^3 = 8i$ gives $w = \left\{2e^{i\pi/6}, 2e^{i5\pi/6}, 2e^{i9\pi/6}\right\}$, or

$$w = \left\{\sqrt{3} + i\,,\ -\sqrt{3} + i\,,\ -2i\right\}$$

Branches of the n-th root function

Can define a **branch** of $z^{1/n}$ by fixing range of $\arg(z)$.

Example 1: Let $\arg_{(-\pi,\pi]}(z)$ be the value of $\arg(z)$ in $(-\pi,\pi]$,

$$\text{define}: \qquad z^{1/2} = |z|^{1/2}\, e^{i \arg_{(-\pi,\pi]}(z)/2}$$

- This choice of square root has positive real part.
- It is not continuous across negative real axis.

Example 2: Let $\arg_{[0,2\pi)}(z)$ be the value of $\arg(z)$ in $[0,2\pi)$,

$$\text{define}: \qquad z^{1/2} = |z|^{1/2}\, e^{i \arg_{[0,2\pi)}(z)/2}$$

- This choice of square root has positive imaginary part.
- It is not continuous across positive real axis.