Lecture 6: Topology of $\mathbb{C}$

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Open subsets of $\mathbb{C}$

Notation

- $\mathbb{C}$ is the set of complex numbers, $\mathbb{R}$ the set of real numbers.
- $D_r(z_0) = \{w : |w - z_0| < r\}$, $\overline{D_r}(z_0) = \{w : |w - z_0| \leq r\}$.

Definition

We say that a set $E \subset \mathbb{C}$ is *open* if, for every $z \in E$, there is some $r > 0$ so that $D_r(z) \subset E$.

- Both the empty set $\emptyset$ and $\mathbb{C}$ are open.
- $D_r(z_0)$ is open (for every r and z_0), but $\overline{D_r}(z_0)$ is not open.
- The complement of $\overline{D_r}(z_0)$, denoted $\mathbb{C} \setminus \overline{D_r}(z_0)$, is open.
- The set $\{z : \mathrm{Im}(z) > 0\}$ is open.

Theorem

- The union of any collection of open sets is open.
- The intersection of a finite collection of open sets is open.

A nonstandard definition

A point z is *strictly inside* E if $D_r(z) \subset E$ for some $r > 0$.

- E is open if every point in E is strictly inside E.

Definition

The interior of E, denoted E^o, is the set of all points that are strictly inside E.

- E^o is the largest open subset of E.
- $D_r(z_0)$ equals the interior of $\overline{D_r}(z_0)$.

Closed sets

Definition

A set $E \subset \mathbb{C}$ is *closed* if $\mathbb{C} \setminus E$ is open.

- Both the empty set $\emptyset$ and $\mathbb{C}$ are closed.
- $\overline{D_r}(z_0)$ is closed.
- The intersection of any collection of closed sets is closed, and the union of a finite collection of closed sets is closed.

Definition

The closure of E, denoted $\overline{E}$, is the set of $z \in \mathbb{C}$ such that $D_r(z)$ intersects E for every $r > 0$.

- $\overline{E}$ is closed for every set E.
- $\overline{E}$ is the smallest closed set containing E.

The boundary of a set

Definition

The boundary of E, denoted ∂E, is $\overline{E} \setminus E^o$.

- $z \in \partial E$ if z in neither strictly inside nor strictly outside E.
- $z \in \partial E$ if $D_r(z)$ intersects both E and $\mathbb{C} \setminus E$ for every $r > 0$.
- For any set E, $\mathbb{C}$ is the disjoint union $E^o \cup \partial E \cup (\mathbb{C} \setminus E)^o$.
- For $r > 0$, $\partial D_r(z) = \partial \overline{D_r}(z) = \{w : |w - z| = r\}$.
- $\partial\{z : \mathrm{Im}(z) > 0\} = \partial\{z : \mathrm{Im}(z) \geq 0\} = \{z : \mathrm{Im}(z) = 0\}$.