Lecture 7: Continuous functions

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Continuous functions in $\mathbb{C}$.

Idea: $f(z)$ continuous if: $|f(w) - f(z)|$ is small if $|w - z|$ is.

Definition

Suppose $E \subset \mathbb{C}$, and $f : E \to \mathbb{C}$. Then f is continuous on E if for all $\epsilon > 0$ and $w \in E$, there exists $\delta > 0$ so that

$$|f(z) - f(w)| < \epsilon \quad \text{when} \quad |z - w| < \delta \quad \text{and} \quad z \in E,$$

equivalently, the image of $D_\delta(w) \cap E$ is contained in $D_\epsilon(f(w))$.

Examples:

- $f(z) = |z|$ is continuous on $\mathbb{C}$.
- $f(z) = z^n$ is continuous on $\mathbb{C}$.
- $f(z) = z^{-1}$ is continuous on $\mathbb{C} \setminus \{0\}$.

Composition of continuous maps

Theorem

Suppose $E, U \subset \mathbb{C}$, f continuous on E, g continuous on U, and $f(E) \subset U$. Then $(g \circ f)(z) = g\big(f(z)\big)$ is continuous on E.

Proof. Assume $z, w \in E$, so $f(z), f(w) \in U$. Then

$$\big|g\big(f(z)\big) - g\big(f(w)\big)\big| < \epsilon \quad \text{if} \quad |f(z) - f(w)| < \delta', \quad \text{some } \delta',$$

$$\text{and} \quad |f(z) - f(w)| < \delta' \quad \text{if} \quad |z - w| < \delta\,, \quad \text{some } \delta\,.$$

Theorem

Suppose $E \subset \mathbb{C}$, and f, g are continuous on E. Then so are $f(z) + g(z)$ and $f(z)g(z)$, and $f(z)/g(z)$ assuming $g(z) \neq 0$.

The principal branch of $\log(z)$ is continuous on $\mathbb{C} \setminus (-\infty, 0]$, that is, on the union $\{\operatorname{Im} z > 0\} \cup \{\operatorname{Im} z < 0\} \cup \{\operatorname{Re} z > 0\}$.

Fact: If $E = \cup_j E_j$ where each E_j is **open**, then f is continuous on E if f is continuous on each E_j.

because: If $w \in E$ then $w \in E_j$, some j. Continuity at w follows by continuity on E_j.

- $\log |z|$ is continuous on $\mathbb{C} \setminus \{0\}$ by the composition rule.
- $$\arg_{(-\pi,\pi]}(z) = \begin{cases} \arctan\big(\operatorname{Im} z \,/\, \operatorname{Re} z\big), & \operatorname{Re} z > 0 \\ \operatorname{arccot}\big(\operatorname{Re} z \,/\, \operatorname{Im} z\big), & \operatorname{Im} z > 0 \\ \operatorname{arccot}\big(\operatorname{Re} z \,/\, \operatorname{Im} z\big) - \pi, & \operatorname{Im} z < 0 \end{cases}$$

The principal branch of $\log(z)$ is **not** continuous on $\mathbb{C} \setminus \{0\}$.

Theorem: assume f is continuous

Suppose that f maps $\mathbb{C}$ to either $\mathbb{C}$ or $\mathbb{R}$. If E is open, then the pre-image $f^{-1}(E) = \{z : f(z) \in E\}$ is an open subset of $\mathbb{C}$.

Proof. Show if $w \in f^{-1}(E)$, then $D_\delta(w) \subset f^{-1}(E)$ some $\delta > 0$.

- Since E open, $E \supset D_\epsilon\big(f(w)\big)$, $\epsilon > 0$.
- f continuous, so $f(z) \in D_\epsilon\big(f(w)\big) \subset E$ if $|w - z| < \delta$.

Example: $\{z : |z^3 + z| < 1\}$ is open.

Fact

Suppose $E \subset \mathbb{C}$ is open, and f is a function from E to $\mathbb{C}$. Then f is continuous if and only if the following property holds:

$f^{-1}(U)$ is an open subset of E whenever U is open.