Lecture 9: Complex differentiation

Hart Smith

Department of Mathematics
University of Washington, Seattle

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Differentiability over $\mathbb{C}$

Assume $E \subset \mathbb{C}$ is open, and f is a function from E to $\mathbb{C}$.

Definition

We say that f is differentiable at a point $w \in E$ if

$$f'(w) = \lim_{z \to w} \frac{f(z) - f(w)}{z - w} \qquad \text{exists.}$$

We say f is analytic on E if it's differentiable at every $w \in E$.

Differentiable at w means: There exists a number $f'(w) \in \mathbb{C}$:

For every $\epsilon > 0$, there exists $\delta > 0$ so that

$$\left| \frac{f(z) - f(w)}{z - w} - f'(w) \right| < \epsilon \quad \text{if} \quad 0 < |z - w| < \delta.$$

Consequences for continuity of f on E

- If f is differentiable at w then f is continuous at w.
- If f is differentiable at w, and for $z \in E$ we define

$$F(z) = \begin{cases} \dfrac{f(z) - f(w)}{z - w}, & z \neq w \\ f'(w), & z = w \end{cases}$$

Then $F(z)$ is continuous at w.

Examples (all defined on $E = \mathbb{C}$)

- $f(z) = 1$ is differentiable at all $w \in \mathbb{C}$, with $f'(w) = 0$.
- $f(z) = z$ is differentiable at all $w \in \mathbb{C}$, with $f'(w) = 1$.
- $f(z) = \bar{z}$ is **not** differentiable at any w.
- $f(z) = e^z$ is differentiable at all $w \in \mathbb{C}$, with $f'(w) = e^w$.

  Proof. Write
  $$\frac{e^{w+\lambda} - e^w}{\lambda} = e^w \left(\frac{e^\lambda - 1}{\lambda} \right)$$
  By the definition of e^λ,
  $$\frac{e^\lambda - 1}{\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{(k+1)!}$$
  which has limit 1 as λ goes to 0.

The usual rules for derivatives hold

Sum and product rules: assume f, g differentiable at w.

Then so are $f+g$ and $f \cdot g$, and

$$(f+g)'(w) = f'(w)+g'(w)\,, \quad (f\cdot g)'(w) = f'(w)g(w)+f(w)g'(w)\,.$$

Chain rule

If g is differentiable at w, and f is differentiable at $g(w)$, then $(f \circ g)(z)$ is differentiable at w, and $(f \circ g)'(w) = f'\big(g(w)\big)\, g'(w)$.

Proof. Take limit as $z \to w$, so $g(z) \to g(w)$, in the expression

$$\frac{f\big(g(z)\big) - f\big(g(w)\big)}{z-w} = \left(\frac{f\big(g(z)\big) - f\big(g(w)\big)}{g(z)-g(w)}\right)\left(\frac{g(z)-g(w)}{z-w}\right)$$

Complex form of Taylor's theorem

Note: $$\left| \frac{f(w+\lambda)-f(w)}{\lambda} - f'(w) \right| = \frac{\left| f(w+\lambda)-f(w)-f'(w)\,\lambda \right|}{|\lambda|}$$

Equivalent definition

f is differentiable at w if there is a complex number $f'(w)$ so

$$f(w+\lambda) = f(w) + f'(w)\,\lambda + r(\lambda), \quad \text{where} \quad \lim_{\lambda\to 0} \frac{|r(\lambda)|}{|\lambda|} = 0.$$

Compare to real differentiability of $\begin{pmatrix} x \\ y \end{pmatrix} \to F(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$

$$F(x+s, y+t) = F(x,y) + DF(x,y) \cdot \begin{pmatrix} s \\ t \end{pmatrix} + r(s,t)$$