Math 427 Final Exam Practice Problem Solutions

1. Find the order of the zero of $f(z)$ at z_0, for:

(a.) $f(z) = z(\sin z)^2\,, \quad z_0 = \pi.$ **Answer** $= 2.$

(b.) $f(z) = (z^2+1)^3\,, \quad z_0 = i.$ **Answer** $= 3.$

(c.) $f(z) = (z^2+4\pi^2)\,(e^z-1)\,, \quad z_0 = 2\pi i.$ **Answer** $= 2.$

2. (a.) Find the radius of convergence of the Taylor expansion of $\dfrac{e^z}{z^2+1}$ about $z_0 = \frac{1}{2}$.

Answer $= \sqrt{\frac{5}{4}} =$ distance of $\frac{1}{2}$ to pole at $z = i$ (or $z = -i$) $=$ radius of largest disc on which the function is analytic.

(b.) Find the values of a_0, a_1, a_2, in the series expansion $\log(1+\sin z) = \sum_{k=0}^{\infty} a_k\, z^k$ about $z_0 = 0$.

Here, log is the principal branch of the logarithm. (You do not need to find the other a_k.)

Answer. Use $a_k = \frac{1}{k!} f^{(k)}(0)$ to find $a_0 = 0$, $a_1 = 1$, $a_2 = -\frac{1}{2}$.

3. Find the isolated singularities of the following functions, and say whether they are removable singularities, poles, or essential singularities.
If one of the singularities is a pole, find the principal part of the function at one of the poles (you can choose which one).

(a.) $\dfrac{z^3}{\sin z}$ **Answer**. Isolated singularities where $\sin z$ has (simple) zeroes, $\{z = k\pi,\ k \in \mathbb{Z}\}$.
At $k = 0$ a removable singularity, at all other k a pole of order 1.
At $z = \pi$ principal part $-\pi^3/(z-\pi)$.

(b.) $\dfrac{e^z}{(z^2+1)^2}$ **Answer**. Poles of order 2 at $z = \pm i$.
At $z = i$, write as $f(z)/(z-i)^2$ with $f(z) = e^z/(z+i)^2$.
Principal part is $f(i)/(z-i)^2 + f'(i)/(z-i)$ (I'll let you figure out the derivative).

(c.) $\dfrac{e^{2z}-1}{z}$ **Answer**. Removable singularity at $z = 0$.

(d.) $z^2 \sin\!\left(\dfrac{1}{z}\right)$ **Answer**. Essential singularity at $z = 0$.

(e.) $\dfrac{\cos z}{(z^2-\pi^2/4)^2}$ **Answer**. Simple poles at $z = \pm\frac{\pi}{2}$. At $z = \frac{\pi}{2}$, write as $f(z)/(z-\frac{\pi}{2})^2$,
where $f(z) = (\cos z)/(z+\frac{\pi}{2})^2$; pole is simple since $f(\frac{\pi}{2}) = 0$.
Principal part is $f'(\frac{\pi}{2})/(z-\frac{\pi}{2}) = -\pi^{-2}/(z-\frac{\pi}{2})$

4. Evaluate the following contour integral; the contour goes counter-clockwise around $\partial D_2(2)$.

$$\int_{|z-2|=2} \frac{e^{\,z}}{(z-1)(z+1)}\, dz$$

Answer. Write as $f(z)/(z-1)$ where $f(z) = e^z/(z+1)$. Use Cauchy integral formula to get $2\pi i f(1) = \pi i e$.