Math 427, Autumn 2017, Homework 1 Solutions

Section 1.1: 5

Solution. Let x, y denote real numbers. Expanding $(x+iy)^2 = i$ gives

$$x^2 - y^2 + 2xy\,i = 0 + 1i \quad \Rightarrow \quad x^2 - y^2 = 0\,, \quad 2xy = 1.$$

$x^2 = y^2$ gives $x = y$ or $x = -y$. When $x = y$, the second equation gives $2x^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$. This leads to square roots of $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$.

If we tried $x = -y$, we would be led to $-2x^2 = 1$, but this does not have any real solutions x.

Section 1.1: 11

Solution. If a is real, and x, y are real, then ax and ay are real. From $a(x+iy) = ax + iay$, we see that ax is the real part of $a(x+iy)$ and ay the imaginary part.

On the other hand, $i(x+iy) = ix - y = (-y) + ix$. So the real and imaginary parts of $i(x+iy)$ are respectively $-y$ and x.

Section 1.1: 15

Solution.

$$\overline{\frac{1}{z}} = \overline{\frac{\overline{z}}{|z|^2}} = \frac{\overline{\overline{z}}}{|z|^2} = \frac{\overline{\overline{z}}}{|\overline{z}|^2} = \frac{1}{\overline{z}}$$

One can also work in (x, y) form:

$$\overline{\frac{1}{x+iy}} = \overline{\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}} = \frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2} = \frac{1}{x-iy}$$

For the other part,

$$1 = \left|z\,\frac{1}{z}\right| = |z|\left|\frac{1}{z}\right| \qquad \text{so} \qquad \frac{1}{|z|} = \left|\frac{1}{z}\right|$$

Section 1.2: 1

Solution. Given some $\epsilon > 0$, we need to find N so that

$$\left|\frac{1}{2+ni} - 0\right| = \frac{1}{|2+ni|} = \frac{1}{\sqrt{4+n^2}} < \epsilon.$$

We can rewrite the last inequality as

$$\sqrt{4+n^2} > \epsilon^{-1} \quad \Leftrightarrow \quad n^2 > \epsilon^{-2} - 4.$$

If we take $N > \epsilon^{-1}$, then $n > N$ implies $n^2 > \epsilon^{-2}$, which gives the desired inequality.

Section 1.2: 2

Solution. The triangle inequality gives

$$|z| \le |z-w| + |w|, \qquad \text{so} \qquad |z| - |w| \le |z-w|$$

Similarly

$$|w| \le |z-w| + |z|, \qquad \text{so} \qquad |w| - |z| \le |z-w|$$

Together these imply $\big||z| - |w|\big| \le |z-w|$.

Section 1.3: 4

Solution.

$$1 + \sqrt{3}\,i \;=\; 2\Big(\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\Big) \;=\; 2\big(\cos\tfrac{\pi}{3} + i\sin\tfrac{\pi}{3}\big)$$

Note that $z \;=\; \log 2 + i\frac{\pi}{3}$ satisfies

$$e^z \;=\; e^{\log 2} e^{i\frac{\pi}{3}} \;=\; 2\big(\cos\tfrac{\pi}{3} + i\sin\tfrac{\pi}{3}\big)$$

So $z \;=\; \log 2 + i\frac{\pi}{3}$ is one solution; all other solutions differ by $i2\pi k$, where k is any integer, so all solutions are given by

$$z \;=\; \log 2 + i\frac{\pi}{3} + i2\pi k\,, \qquad \text{for some integer } k\,.$$

Section 1.3: 6

Solution. Let $z = x + iy$. We have $e^{x+iy} = e^x e^{iy} = e^x \cos y + ie^x \sin y$. Then

$$|e^z| = \big(e^x \cos y\big)^2 + \big(e^x \sin y\big)^2 = e^x(\cos^2 y + \sin^2 y) = e^x = e^{\mathrm{Re}(z)}.$$

Also, $\mathrm{Re}(z) \le |\mathrm{Re}(z)| \le |z|$, and e^x is increasing in x, so

$$e^{\mathrm{Re}(z)} \le e^{|z|}$$

Section 1.3: 9

Solution. $e^z = e^x \cos y + ie^x \sin y$ is real if and only if $\sin y = 0$ (since $e^x \ne 0$). This happens if $y = \pi k$, for k an integer, so

$$e^z \;\text{ is real if and only if }\; \mathrm{Re}(z) = k\pi.$$

Similarly, $e^z = e^x \cos y + ie^x \sin y$ is pure imaginary when $\cos y = 0$, so

$$e^z \;\text{ is pure imaginary if and only if }\; \mathrm{Im}(z) = \frac{\pi}{2} + k\pi.$$

Additional Problem 1:
Solution.

$$|z| = |1+3i| = \sqrt{1+9} = \sqrt{10}$$

$$|w| = |2-5i| = \sqrt{4+25} = \sqrt{29}$$

$$|z+w| = |3-2i| = \sqrt{9+4} = \sqrt{13}$$

Check that $\sqrt{10}+\sqrt{29} \geq \sqrt{13}$ easily since $\sqrt{29} > \sqrt{13}$.

Additional Problem 2.1:
Solution. The modulus of the ratio of the $j+1$ term to the j term is

$$\left| \frac{\frac{z^{j+1}}{2(j+1)}}{\frac{z^j}{2j}} \right| = \frac{2j}{2j+2}\,|z|$$

The limit as $j \to \infty$ of this is $|z|$, so the ratio test gives convergence if $|z| < 1$.

Additional Problem 2.2:
Solution. The modulus of the ratio of the $j+1$ term to the j term is

$$\left| \frac{\frac{z^{2(j+1)+1}}{2^{j+1+1}}}{\frac{z^{2j+1}}{2^{j+1}}} \right| = \frac{1}{2}|z|^2$$

This is constant in j, so the limit is $\frac{1}{2}|z|^2$, and the ratio test gives convergence if $\frac{1}{2}|z|^2 < 1$ or $|z| < \sqrt{2}$.