Math 427, Autumn 2019, Homework 2 Solutions

Section 1.4: 3

Solution. The powers take the form $e^{\frac{in\pi}{8}}$, for n an integer. We have

$$e^{\frac{in\pi}{8}} = e^{\frac{im\pi}{8}} \quad \Leftrightarrow \quad \frac{n\pi}{8} - \frac{m\pi}{8} = 2\pi k$$

for some integer k. This happens if $m = n + 16k$ for some integer k. So considering $n = 0, \dots, 15$ gives all possible distinct powers, since every number has a unique representation as $n + 16k$ for integers k and n with $0 \le n \le 15$.

Section 1.4: 8

Solution. Write $-9 = 9e^{i\pi}$, which is the polar form. Then in polar form the cube roots are (where $9^{\frac{1}{3}}$ is the positive cube root)

$$9^{\frac{1}{3}}e^{i\pi/3}, \;\; 9^{\frac{1}{3}}e^{i\pi}, \;\; 9^{\frac{1}{3}}e^{i5\pi/3}.$$

Using $\cos(\frac{\pi}{3}) = \frac{1}{2}$, $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$, we can write these as

$$\frac{9^{\frac{1}{3}}}{2} + i\frac{9^{\frac{1}{3}}\sqrt{3}}{2}, \quad -9^{\frac{1}{3}}, \quad \frac{9^{\frac{1}{3}}}{2} - i\frac{9^{\frac{1}{3}}\sqrt{3}}{2}.$$

Section 1.4: 15

Solution. Let $\log(z) = \log|z| + i\arg_{(-\pi,\pi]}(z)$ be the principal branch.

$$z^i = e^{i\log(z)} = e^{i\log|z|}\, e^{-\arg_{(-\pi,\pi]}(z)}$$

The term $e^{i\log|z|}$ is continuous on $\mathbb{C}\backslash\{0\}$, but the term $e^{-\arg_{(-\pi,\pi]}(z)}$ jumps as z crosses the negative real axis. Precisely, if z is on the negative real axis, say $z = -x$ where $x > 0$, then $\arg_{(-\pi,\pi]}(z) = \pi$, and

$$(-x)^i = e^{i\log(x)}e^{-\pi}\,.$$

If we approach $-x$ by z from below, then $\arg_{(-\pi,\pi]}(z)$ approaches $-\pi$, and the limit from below would be $e^{i\log(x)}e^{\pi}$.

Section 1.4: 16

Solution. Since the principal branch of $\sqrt{\cdot}$ is continuous on $\mathbb{C}\backslash(-\infty,0]$, the function $\sqrt{1-z^2}$ is continuous on the set $1 - z^2 \notin (-\infty, 0]$. This is the same as $z^2 \notin [1,\infty)$, which is the same as $z \notin (-\infty,-1] \cup [1,\infty)$. So, the branch is continuous on $\mathbb{C}\backslash\{(-\infty,-1] \cup [1,\infty)\}$. On the other hand, the branch has a jump at each point in $(-\infty,-1) \cup (1,\infty)$. To see this for $z = x \in (1,\infty)$, note that $1 - x^2$ is on the negative real axis, so the principal branch of $\sqrt{1-x^2} = i|1-x^2|^{\frac{1}{2}}$, that is, it lies on the positive imaginary axis.

We then consider taking $z = x + iy$ for $x > 1$ and $y > 0$ and letting y got to 0. Then $1 - z^2 = 1 - x^2 + y^2 - i2xy$ has imaginary part less than 0. So the principal branch of $\sqrt{\cdot}$ assigns it a root near the negative imaginary axis. Precisely, $\lim_{y\to 0} \arg(1 - x^2 + y^2 - i2xy) = -\pi$. So the principal branch of the square root gives

$$\lim_{y\to 0} \sqrt{1 - x^2 + y^2 - i2xy} \;=\; -i|1 - x^2|^{\frac{1}{2}}.$$

Section 2.1: 4

Solution. Suppose $w \in A$. Letting $r = \mathrm{Re}(w) > 0$, we will show that $D_r(w) \subset A$. If $|z - w| < r$, the $|\mathrm{Re}(z) - \mathrm{Re}(w)| < \mathrm{Re}(w)$, hence

$$0 < \mathrm{Re}(z) < 2\mathrm{Re}(w)$$

which implies that $z \in A$.

Section 2.1: 5

Solution. (a) is open, (b) is neither open nor closed, (c) is closed.

Section 2.1: 7

Solution. We use Theorem 2.1.6(b). If every neighborhood of z contains a point of E, then we can choose $z_n \in D_{\frac{1}{n}}(z) \cap E$, that is $z_n \in E$ and $|z_n - z| < \frac{1}{n}$. Then $\{z_n\}$ is a sequence in E that converges to z. Conversely, if $\{z_n\}$ is a sequence in E that converges to z, then every neighborhood of z will contain some z_n (in fact infinitely many z_n), hence intersects E. This is because every neighborhood contains $D_r(z)$ for some $r > 0$, but since $\{z_n\}$ converges to z we must have $|z_n - z| < r$ for $n > N$, for some N depending on r.

Section 2.1: 14

Solution. Use Theorem 2.1.13. The set of z such that $|f(z)| < r$ is the preimage under f of the open set $\{w : |w| < r\}$, hence is open, and $\mathrm{Re}(f(z)) < r$ is the preimage under f of the open set $\{w : \mathrm{Re}(w) < r\}$.

Additional problem.

Solution. This is similar to Problem 1.4 #16 above. The branch of the square root defined using $\arg_{(-\pi,\pi]}$ will be continuous on $\mathbb{C}\backslash(-\infty, 0]$. So $\sqrt{z^2 + 1}$ is continuous where $z^2 + 1 \notin (-\infty, 0]$ or $z^2 \notin (-\infty, -1]$ or $z \notin (-i\infty, -i] \cup [i, i\infty)$. It has jumps at all points in $(-i\infty, -i) \cup (i, i\infty)$.