Math 427, Autumn 2019, Homework 3 Solutions

Section 2.2: 8

Solution. The principal branch of $\log z$ is differentiable (as shown in lecture) on the set $\mathbb{C}\backslash(-\infty,0]$; that is, the complement of the cut line, and $(\log z)' = z^{-1}$. The function z is differentiable everywhere on $\mathbb{C}$, with derivative 1, and is non zero on $\mathbb{C}\backslash\{0\}$. In particular it is non zero on the set $\mathbb{C}\backslash(-\infty,0]$. Consequently the ratio $(\log z)/z$ is differentiable on $\mathbb{C}\backslash(-\infty,0]$, and we can use the ratio rule to find its derivative there

$$\left(\frac{\log z}{z}\right)' = \frac{z^{-1}\cdot z - \log z}{z^2} = \frac{1-\log z}{z^2}$$

Section 2.3: 5

Solution.

(a.) $\gamma(t) = z_0 + re^{it}, \quad t \in [0,2\pi]$.

(b.) $\gamma(t) = z_0 + re^{-it}, \quad t \in [0,2\pi]$.

(c.) $\gamma(t) = z_0 + re^{it}, \quad t \in [0,6\pi]$.

Section 2.3: 10

Solution. Let $\gamma(t) = 3e^{it}\,,\ t \in [0,2\pi]\,,$ so $\gamma'(t) = i3e^{it}\,.$

$$\int_\gamma z^{-1}\,dz \;=\; \int_0^{2\pi} \big(3e^{it}\big)^{-1}\, i3e^{it}\,dt \;=\; \int_0^{2\pi} i\,dt \;=\; 2\pi i\,.$$

$$\int_\gamma \overline{z}\,dz \;=\; \int_0^{2\pi} \overline{3e^{it}}\; i3e^{it}\,dt \;=\; \int_0^{2\pi} 9i\,dt \;=\; 18\pi i\,.$$

Additional Problem 1:

(a).
$$\sin(x+iy) = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \frac{e^{-y}e^{ix} - e^{y}e^{-ix}}{2i}$$
$$= \frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)}{2i}$$
$$= \left(\frac{e^{y}+e^{-y}}{2}\right)\sin x + i\left(\frac{e^{y}-e^{-y}}{2}\right)\cos x$$

$$u(x,y) = \sin x\,\cosh y\,, \qquad v(x,y) = \cos x\,\sinh y.$$

(b). $$(x+iy)^2+(x-iy)^2 = (x^2-y^2+2ixy)+(x^2-y^2-2ixy) = 2x^2-2y^2$$

$$u(x,y) = 2x^2 - 2y^2\,, \qquad v(x,y) = 0.$$

(c). $$(x+iy)e^{-x-iy} = (x+iy)e^{-x}(\cos y - i\sin y)$$
$$= e^{-x}(x\cos y + y\sin y) + ie^{-x}(y\cos y - x\sin y)$$

$$u(x,y) = e^{-x}(x\cos y + y\sin y)\,, \qquad v(x,y) = e^{-x}(y\cos y - x\sin y)$$

(d). $$|x+iy|^2 = x^2+y^2$$

$$u(x,y) = x^2 + y^2\,, \qquad v(x,y) = 0.$$

Additional Problem 2:

(a.) Yes: $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = \cos x\, \cosh y\,, \quad \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} = -\sin x\, \sinh y\,.$

(b.) No: $\dfrac{\partial u}{\partial x} = 4x\,, \quad \dfrac{\partial v}{\partial y} = 0$

(c.) Yes:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^{-x}(-x\cos y - y\sin y + \cos y)$$
$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^{-x}(-y\cos y + x\sin y - \sin y)$$

(d.) No: $\dfrac{\partial u}{\partial x} = 2x\,, \quad \dfrac{\partial v}{\partial y} = 0$

Additional Problem 3: Where $\log z$ is differentiable we differentiate both sides of the identity $e^{\log z} = z$ to get $e^{\log z}\cdot(\log z)' = 1$ which, setting $e^{\log z} = z$ again, gives $z(\log z)' = 1$. So, wherever the branch of $\log z$ is differentiable, we must have $(\log z)' = 1/z$.