Math 427, Autumn 2019, Homework 4 Solutions

Section 2.4: 2

Solution. We get the same answer for any parameterization of the path, so we choose the most convenient one $\gamma(t) = e^{it}$, $t \in [0, 4\pi]$. Then

$$\int_\gamma \frac{1}{z}\,dz = \int_0^{4\pi} \frac{ie^{it}}{e^{it}}\,dt = \int_0^{4\pi} i\,dt = 4\pi i$$

Section 2.4: 8

Solution. We first show that $|\cos z| \le e$ in $|z| = 1$, as follows:

$$|\cos z| = \left|\frac{e^{iz} + e^{-iz}}{2}\right| \le \frac{|e^{iz}| + |e^{-iz}|}{2} = \frac{e^{-y} + e^{y}}{2} \le e \ \text{ if } \ |z| \le 1.$$

When $|z| = 1$, this gives $\left|\dfrac{\cos z}{z}\right| = \dfrac{|\cos z|}{|z|} \le \dfrac{e}{1} = e.$

We now apply Theorem 2.4.9, and the fact that $\ell(\partial D_1(0)) = 2\pi$, to prove the estimate.

Section 2.4: 10

Solution. Let $p(z) = \sum_{k=0}^{n} a_k\, z^k$. Then

$$\int_\gamma p(z)\,dz \;=\; \int_0^{2\pi} \sum_{k=0}^{n} a_k\,(e^{it})^k\, ie^{it}\,dt \;=\; i\sum_{k=0}^{n} a_k \int_0^{2\pi} e^{i(k+1)t}\,dt\,.$$

Since $k + 1 > 0$, we have

$$\int_0^{2\pi} e^{i(k+1)t}\,dt \;=\; \tfrac{1}{i(k+1)}\, e^{i(k+1)t}\Big|_0^{2\pi} \;=\; \tfrac{1}{i(k+1)}\left(e^{i(k+1)2\pi} - 1\right) \;=\; 0\,.$$

Section 2.6: 2

Solution. The function $(z^2-4)^{-1}$ is analytic on $\mathbb{C}\backslash\{-2, 2\}$, in particular it is analytic on the convex set $\{z : |z| < 2\}$. Since γ is a closed path contained in the set $\{z : |z| < 2\}$, then

$$\int_\gamma (z^2-4)^{-1}\,dz \;=\; 0\,.$$

Section 2.6: 5

Solution. $1/z^2$ has an anti-derivative on $\mathbb{C}\backslash\{0\}$: $\dfrac{1}{z^2} = \left(-\dfrac{1}{z}\right)'$.

So for any path closed path γ in $\mathbb{C}\backslash\{0\}$:

$$\int_\gamma \frac{1}{z^2}\,dz \;=\; -\frac{1}{z}\Big|_{\gamma(a)}^{\gamma(b)} \;=\; 0\,, \quad \text{since} \quad \gamma(b) = \gamma(a)\,.$$