Math 427, Autumn 2019, Homework 6 Solutions

Section 3.2: 1

Solution. The power series expansion of $(1-z)^{-1}$ about 0 is the geometric series:

$$(1-z)^{-1} = \sum_{k=0}^{\infty} z^k$$

which converges for $|z| < 1$. Since $(1-z)^{-2}$ is the derivative of $(1-z)^{-1}$, and we can differentiate power series term by term, we have

$$(1-z)^{-2} = \sum_{k=1}^{\infty} k\, z^{k-1} = \sum_{k=0}^{\infty} (k+1)\, z^k$$

Section 3.2: 9

Solution. We did this in Lecture 24.

Section 3.2: 10

Solution. Suppose $0 < r < R$. Then f is analytic on an open set containing the closed disc $D_r(z_0)$. We can apply Cauchy's estimates to see $|f'(z_0)| \leq M/r$. This is true for every $r < R$, so we must have $|f'(z_0)| \leq M/R$ by taking limits, for example.

Section 3.3: 2

Solution. We have that $\lim_{z\to\infty} 1/f(z) = 0$ if for all $\epsilon > 0$ there is M such that $|1/f(z)| < \epsilon$ if $|z| > M$. This is equivalent to the statement: for all $\epsilon > 0$ there is M such that $|f(z)| > 1/\epsilon$ if $|z| > M$, and letting $\epsilon = 1/K$ (assuming $K \neq 0$, or take $\epsilon = 1$ if $K = 0$): for all K there is M such that $|f(z)| > K$ if $|z| > M$.

Section 3.3: 3

Solution. Supose by contradiction that $f(z) \neq 0$ for every $z \in C$ and $\lim_{z\to\infty} f(z) = \infty$. Let $g(z) = 1/f(z)$. Then $g(z)$ is entire, and $\lim_{z\to\infty} g(z) = 0$. By Liouville's theorem $g(z)$ is constant, and the constant must be 0 since the limit at ∞ is 0.

Section 3.3: 4

Solution. Suppose that $|f(z)| \geq 1$ for all $z \in \mathbb{C}$. Then the function $f(z)^{-1}$ is an entire function, since $f(z) \neq 0$ for all z, and we have $|f(z)^{-1}| = |f(z)|^{-1} \leq 1$ for all z. By Liouville's Theorem, $f(z)^{-1}$ is constant, so $f(z)$ is constant.

Section 3.4: 3

Solution. The first answer is no. If $f(z)$ is analytic on $\mathbb{C}$, and $f(\frac{1}{n}) = 0$ for all integers $n \geq 1$, then $f(0) = 0$. If f is not identically zero, this would contradict Theorem 3.4.2 (b), since any neighborhood of 0 contains points of the form $\frac{1}{n}$. Alternatively, one can apply Theorem 3.4.4 and use that $f(z)$ equals the 0 function a non-discrete set.
For the second part, there is a non-zero function on $\mathbb{C}\backslash\{0\}$ which has a zero at $z = \frac{1}{n}$ for every $n \geq 1$. For example, $\sin(\pi/z)$.

Section 3.4: 4

Solution. Write

$$\sin(z) - z = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!}\, z^{2k+1} = -\frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \cdots$$

Then we see that $\sin(z) - z$ has a zero of order 3, and

$$\sin(z) - z = z^3 \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!}\, z^{2k-2} = z^3 \Big(-\frac{1}{3!} + \frac{1}{5!}z^2 - \frac{1}{7!}z^4 + \cdots \Big)$$

Additional problem 1.

Solution. The function $\tan(z)$ is odd: $\tan(-z) = -\tan z$. So if we expand

$$\tan z = \sum_{k=0}^{\infty} a_k\, z^k$$

then

$$\tan(-z) = \sum_{k=0}^{\infty} a_k\, (-z)^k = \sum_{k=0}^{\infty} (-1)^k a_k\, z^k.$$

This must equal

$$-\tan(z) = \sum_{k=0}^{\infty} (-a_k)\, z^k.$$

So $-a_k = (-1)^k a_k$. If k is even then $-a_k = a_k$, so $a_k = 0$.

Additional problem 2.

Solution. Differentiate $\tan z = \dfrac{\sin z}{\cos z}$ to find

$$a_1 = (\tan z)'(0) = 1, \qquad a_3 = \frac{1}{3!}(\tan z)'''(0) = \frac{1}{3!} \cdot 2 = \frac{1}{3}.$$