Math 427, Autumn 2019, Homework 7 Solutions

Section 3.4: 12

Solution. We factor

$$\frac{1}{z-z^3} = (-1)\,\frac{1}{z}\,\frac{1}{z-1}\,\frac{1}{z+1}$$

There are simple poles (i.e poles of order 1) at $z = 0,\ 1,\ -1$.

Section 3.4: 13

Solution. $\sin(1/z)$ has an isolated singularity at $z = 0$, which is an essential singularity. One can see that it is essential since $\lim_{z\to 0}\sin(1/z)$ does not exist. (Consider z real, for example, to see this.)

Section 3.4: 14

Solution.

$$\frac{e^z - z - 1}{z^2} = \frac{\frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots}{z^2} = \tfrac{1}{2!} + \tfrac{1}{3!}z + \cdots$$

So the singularity is removable, and $f(0) = \frac{1}{2!} = \frac{1}{2}$.

Section 3.4: 15

Solution. This one is more complicated. The function $\dfrac{1}{z}$ has a simple pole at $z = 0$. The function $\dfrac{1}{e^z - 1}$ has simple poles at $z = k2\pi i$, for $k = 0, \pm 1, \pm 2, \ldots$, since $e^z - 1$ has zeroes of order 1 at those points.
From this, we can see that the difference $\dfrac{1}{e^z-1} - \dfrac{1}{z}$ has a simple pole at $z = k2\pi i$ for $k = \pm 1, \pm 2\ \ldots$, since the factor $\dfrac{1}{z}$ is analytic at those points, so the principal part of the function at those points is the same as the principal part of $\dfrac{1}{e^z-1}$.
To see whether 0 is a pole or a removable singularity, we can write

$$\frac{1}{e^z-1} - \frac{1}{z} = (-1)\frac{e^z - z - 1}{z(e^z-1)}$$

Both the numerator and denominator have zeroes of order 2 at $z = 0$, so the fraction has a removable singularity. One could also use l'Hopital's rule to see that

$$\lim_{z\to 0}\frac{e^z - z - 1}{z(e^z-1)} = \frac{1}{2}$$

and since the limit exists it is a removable singularity.

Section 3.4: 16

Solution. The function $\dfrac{\log z}{(z-1)^2}$ has domain $\mathbb{C}\backslash\{(-\infty,0]\cup\{1\}\}$, and an isolated singularity at $z=1$. To find the nature of the singularity at $z=1$, we expand $\log z$ about $z=1$ in terms of its Taylor expansion:

$$\begin{aligned}\log(z) &= \log(1) + \log'(1)(z-1) + \frac{1}{2!}\log''(z)(z-1)^2 + \cdots \\ &= (z-1) - \frac{1}{2}(z-1)^2 + \cdots\end{aligned}$$

Since this vanishes to first order, there is a pole of order 1 at $z=1$. One can find the principal part (though that is not part of the assigned problem):

$$\frac{\log z}{(z-1)^2} = \frac{(z-1) - \frac{1}{2}(z-1)^2 + \cdots}{(z-1)^2} = \frac{1}{z-1} - \frac{1}{2} + \cdots$$

so the principal part is $\dfrac{1}{z-1}$.

Extra problem 1.

Solution. We factor $z^3-1=(z-1)(z-e^{2\pi i/3})(z-e^{4\pi i/3})$. We then equate

$$\frac{1}{(z-1)(z-e^{2\pi i/3})(z-e^{4\pi i/3})} = \frac{a}{z-1} + \frac{b}{z-e^{2\pi i/3}} + \frac{c}{z-e^{4\pi i/3}}$$

By plugging in roots, or comparing principal parts, we see that

$$a = \frac{1}{(1-e^{2\pi i/3})(1-e^{4\pi i/3})}, \quad b = \frac{1}{(e^{2\pi i/3}-1)(e^{2\pi i/3}-e^{4\pi i/3})},$$
$$c = \frac{1}{(e^{4\pi i/3}-1)(e^{4\pi i/3}-e^{2\pi i/3})}$$

Extra problem 2.

Solution. When $f(z)=\sum_{k=-2}^{\infty} a_k(z-z_0)^k$ we can write

$$(z-z_0)^2 f(z) = \sum_{k=0}^{\infty} a_{k-2}(z-z_0)^k.$$

The right hand side is a convergent power series, so it is analytic, and in particular continuous. So we can calculate the limit by evaluating the series at $z=z_0$:

$$\lim_{z\to z_0}(z-z_0)^2 f(z) = a_{-2}$$