## Math 427 Midterm, Autumn 2016, Solutions to 4 and 5

4. 

(a.) Find all complex numbers $z$ such that $z^{3}=1-i$. Express the answers in the form $x+i y$ (i.e. don't leave them in polar form). The answers for $x$ and $y$ can be given using sin and cos functions.
Solution. Write $1-i=\sqrt{2} e^{-i \frac{\pi}{4}}$, and use rule for roots to get

$$
z=2^{\frac{1}{6}} e^{-i \frac{\pi}{12}}, \quad 2^{\frac{1}{6}} e^{-i \frac{\pi}{12}+i \frac{2 \pi}{3}}, \quad 2^{\frac{1}{6}} e^{-i \frac{\pi}{12}+i \frac{4 \pi}{3}} .
$$

In $x+i y$ form, these take the form $2^{\frac{1}{6}} \cos \left(\frac{\pi}{12}\right)-i 2^{\frac{1}{6}} \sin \left(\frac{\pi}{12}\right), 2^{\frac{1}{6}} \cos \left(\frac{7 \pi}{12}\right)+i 2^{\frac{1}{6}} \sin \left(\frac{7 \pi}{12}\right)$, $2^{\frac{1}{6}} \cos \left(\frac{15 \pi}{12}\right)+i 2^{\frac{1}{6}} \sin \left(\frac{15 \pi}{12}\right)$.
(b.) Let $0<r<R$ be real numbers. Describe the set of $z \in \mathbb{C}$ such that $r<\left|e^{z}\right|<R$.

Solution. Equivalent to $r<e^{\operatorname{Re}(z)}<R$ or $\log r<\operatorname{Re}(z)<\log R$.
(c.) Describe the image $f(E)$ where $E$ is the set $\{z \in \mathbb{C}: \operatorname{Im}(z)>0$ and $\operatorname{Re}(z)>0\}$, and $f(z)=z^{3}$. As part of your answer, sketch the image in the complex plane. Indicate whether any boundary lines are included or not in the image.
Solution. The cube of the open first quadrant is the first 3 quadrants, or the union of sets $\{z: \operatorname{Im}(\mathrm{z})>0\} \cup\{z: \operatorname{Re}(\mathrm{z})<0\}$.
5. Express the following functions in the form $f(x+i y)=u(x, y)+i v(x, y)$ for real-valued functions $u$ and $v$. That is, give explicit formulas for $u(x, y)$ and $v(x, y)$.
(a.) $f(z)=z \log z$, on the set $\operatorname{Re}(z)>0$, where $\log z$ is the principal branch.

## Solution.

$$
\begin{aligned}
& (x+i y)\left(\log \left(\sqrt{x^{2}+y^{2}}\right)+i \arctan \left(\frac{y}{x}\right)\right) \\
& \quad=\frac{1}{2} x \log \left(x^{2}+y^{2}\right)-y \arctan \left(\frac{y}{x}\right)+i\left(x \arctan \left(\frac{y}{x}\right)+\frac{1}{2} y \log \left(x^{2}+y^{2}\right)\right)
\end{aligned}
$$

(b.) $f(z)=\frac{e^{z}}{z}$, on the set $z \neq 0$.

Solution. $e^{x}(\cos y+i \sin y) \frac{x-i y}{x^{2}+y^{2}}=\frac{e^{x}(x \cos y+y \sin y)}{x^{2}+y^{2}}+i \frac{e^{x}(x \sin y-y \cos y)}{x^{2}+y^{2}}$

