Lecture 1: Schwarz's Lemma

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Assume that: $f(z)$ is analytic on $D_1(0) = \{z : |z| < 1\}$,

and continuous on $\overline{D_1}(0) = \{z : |z| \leq 1\}$.

By the Maximum Modulus Theorem:

- If $|f(z)| \leq 1$ when $|z| = 1$, then $|f(z)| \leq 1$ when $|z| \leq 1$,
- $|f(z)| < 1$ when $|z| < 1$ unless $f(z)$ is constant.

Example: the function $f(z) = \dfrac{z + \frac{1}{2}}{\frac{1}{2}z + 1}$

If $|z| = 1$, then $z\overline{z} = 1$, so:

$$|f(z)| = \left|\frac{z + \frac{1}{2}}{\frac{1}{2}z + z\overline{z}}\right| = \frac{1}{|z|}\left|\frac{z + \frac{1}{2}}{\frac{1}{2} + \overline{z}}\right| = 1$$

Therefore: $|f(z)| < 1$ if $|z| < 1$, $|f(z)| = 1$ if $|z| = 1$.

Theorem: assume f analytic on $D_1(0)$, continuous on $\overline{D_1}(0)$.

Suppose that $|f(z)| = 1$ when $|z| = 1$. If $f(z)$ is not constant, then there is some point $z \in D_1(0)$ where $f(z) = 0$.

Proof. By Maximum Modulus, $|f(z)| < 1$ when $|z| < 1$.

- If $f(z) \neq 0$ on $D_1(0)$, then $1/f(z)$ is analytic, continuous.
- By assumption, $|1/f(z)| = 1/|f(z)| = 1$ if $|z| = 1$.
- Max Mod implies $1/|f(z)| < 1$ if $|z| < 1$, a contradiction.

Stronger fact: if $|w| < 1$, then $w = f(z)$ for some $|z| < 1$.

Typical such map: $f(z) = z^n, \;\; n \geq 1$.

Schwarz's Lemma

Assume that $f(z)$ is analytic on $D_1(0)$, and $|f(z)| \leq 1$ for $|z| < 1$.
If $f(0) = 0$, then $|f(z)| \leq |z|$ for all $|z| < 1$, and $|f'(0)| \leq 1$.
If $|f'(0)| = 1$, or $|f(z)| = |z|$ some z, then $f(z) = cz$, $|c| = 1$.

Proof. The function $g(z) = \begin{cases} f(z)/z\,, & 0 < |z| < 1, \\ f'(0)\,, & z = 0, \end{cases}$

is analytic on $D_1(0)$. For every $r < 1$:

$$\text{if} \quad |z| = r: \quad |g(z)| = |f(z)|/|z| \leq 1/r.$$

By Max Mod: $|g(z)| \leq r^{-1}$ if $|z| < r$. This holds for all $r < 1$, so

$$|g(z)| \leq 1 \quad \text{if} \quad |z| < 1.$$

If $|g(z)| = 1$ for some $|z| < 1$, i.e. $|f'(0)| = 1$ or $|f(z)| = |z|$,

$$\text{by Max Mod} \quad g(z) = c\,, \quad \text{so} \quad f(z) = cz.$$

Bi-analytic maps of $D_1(0)$

Theorem

Assume $f(z)$ is a 1-1 map of $D_1(0)$ onto $D_1(0)$, and f and f^{-1} are analytic functions. If $f(0) = 0$, then $f(z) = cz$, with $|c| = 1$.

Proof. Schwarz's lemma applies to both $f(z)$ and $f^{-1}(z)$:

- $f(0) = 0$ so $|f'(0)| \leq 1$, and $f^{-1}(0) = 0$ so $|(f^{-1})'(0)| \leq 1$.
- Differentiate $f(f^{-1}(z)) = z$: by chain rule $f'(0)(f^{-1})'(0) = 1$.
- Conclude $|f'(0)| = 1$. By Schwarz, $f(z) = cz$ with $|c| = 1$.

Result fails if $f(z)$ is not 1-1. Example: $f(z) = z^2$